COUNTEREXAMPLE TO WINKLER'S CONJECTURE ON VENN DIAGRAMS

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ABSTRACT. In 1984, Peter Winkler conjectured that every simple Venn diagram with n curves can be extended to a simple Venn diagram with $n+1$ curves. We present a counterexample to his conjecture for $n = 7$, which is obtained by combining theoretical ideas with computer assistance from state-of-the-art SAT solvers.

1. INTRODUCTION

Venn diagrams are a popular tool to illustrate the relations between sets and operations on them, such as union and intersection, or the well-known inclusion-exclusion principle. Historically, these diagrams were introduced by John Venn [Ven80] (1834–1923) in the context of formal logic. Most of these illustrations depict the Venn diagram with three sets represented by unit circles; see Figure 1 (a). It may even come to a surprise that Venn diagrams exist for any number of sets, not just three. Formally, an *n-Venn diagram* is a collection of n simple closed curves in the plane that intersect in only finitely many points and create exactly 2^n regions, one for every possible combination of being inside or outside of each of the n curves. Figures 1 (b)+(c) display diagrams with $n = 4$ and $n = 5$ curves, respectively.

A Venn diagram is *simple* if at most two of the n curves intersect in any point: The three diagrams in the top row of Figure 1 are simple, whereas the three diagrams in the bottom row are non-simple. A simple Venn diagram has exactly $2^n - 2$ crossings, whereas non-simple diagrams may have much fewer. For humans, simple Venn diagrams tend to be easier to read than non-simple diagrams.

1.1. Winkler's conjecture.

In 1984, Peter Winkler [Win84] raised the following intriguing conjecture.

Conjecture 1. *Every simple n-Venn diagram can be extended to a simple $(n+1)$-Venn diagram by the addition of a suitable curve.*

He reiterated the problem in his 'Puzzled' column [Win12] in the Communications of the ACM. The conjecture is also listed as the first open problem in Ruskey and Weston's wonderful survey [RW97] on Venn diagrams.

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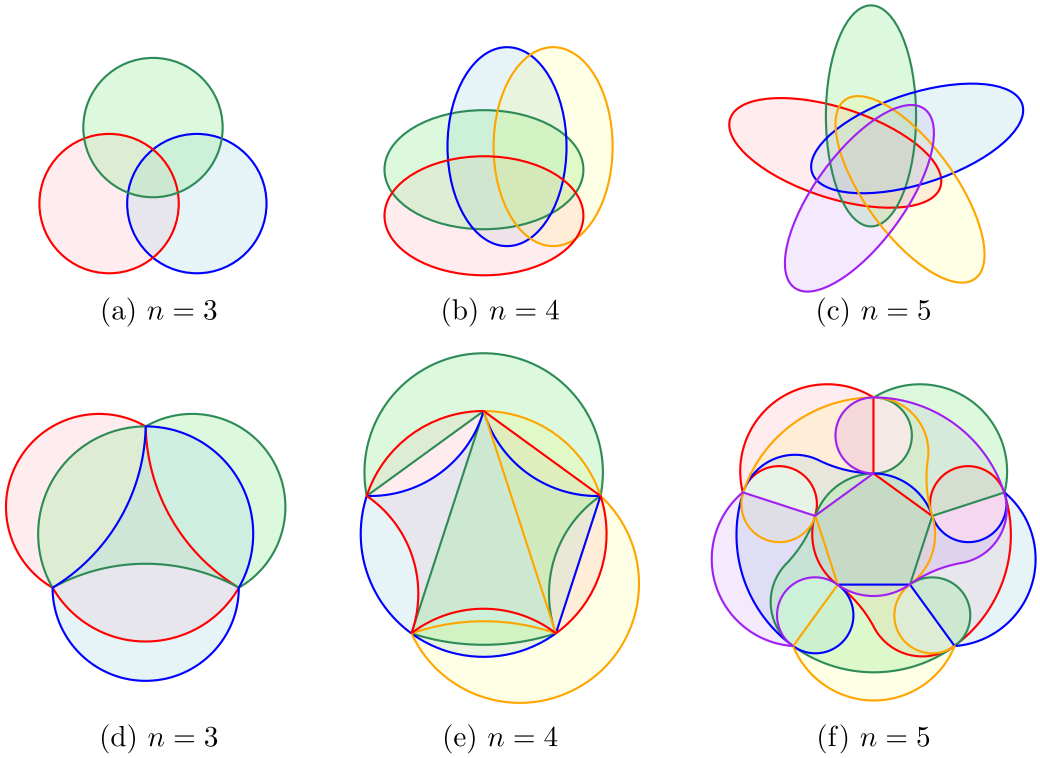

FIGURE 1. Venn diagrams with $n = 3, 4, 5$ curves: (a), (b), (c) are simple, whereas (d), (e), (f) are non-simple; (a), (b) and (d) are reducible, whereas (c), (e) and (f) are irreducible; (a), (b), (c) and (d) are monotone, whereas (e) and (f) are not monotone.

Concerning small cases, the conjecture is easily verifiable for all simple diagrams on $n \in \{3, 4, 5\}$ curves, simply because there are so few of them. In fact, for $n = 3$ and $n = 4$ there is only one simple Venn diagram, and for $n = 5$ there are only 20 simple Venn diagrams [HP97, CHP00]. These counts are up to stereographic projection that makes any region the outer region; or equivalently, by considering the diagrams on a sphere. The number of n-Venn diagrams for $n \geq 6$ is unknown.

Another class of Venn diagrams for which the conjecture is easily seen to be true are reducible diagrams. An n-Venn diagram is *reducible* if one of its curves can be removed so that the remaining $n - 1$ curves form an $(n - 1)$-Venn diagram. The unique simple diagrams on $n = 3$ and $n = 4$ curves are reducible, and among the 20 simple diagrams with $n = 5$ curves, 11 are reducible and 9 are irreducible (the diagram in Figure 1 (c) is irreducible). If a diagram is reducible, then the curve C that can be removed splits every region of the resulting $(n - 1)$-Venn diagram into two, and therefore this curve touches every region in the n-Venn diagram. We can thus add an $(n + 1)$st curve by following the curve C and subdividing every region into two, changing sides along C exactly once along every segment between two consecutive crossings. Thus, the interesting instances of Conjecture 1 are irreducible diagrams.

Grünbaum [Grü92] modified Winkler's conjecture, by dropping the requirement for the diagrams to be simple. This makes the problem harder in the sense that more diagrams are allowed, but easier in the sense that there is substantially more freedom when adding the new curve. This variant of the conjecture was proved subsequently by Chilakamarri, Hamburger and Pippert [CHP96], by using a straightforward reduction to a classical result of Whitney [Whi31].

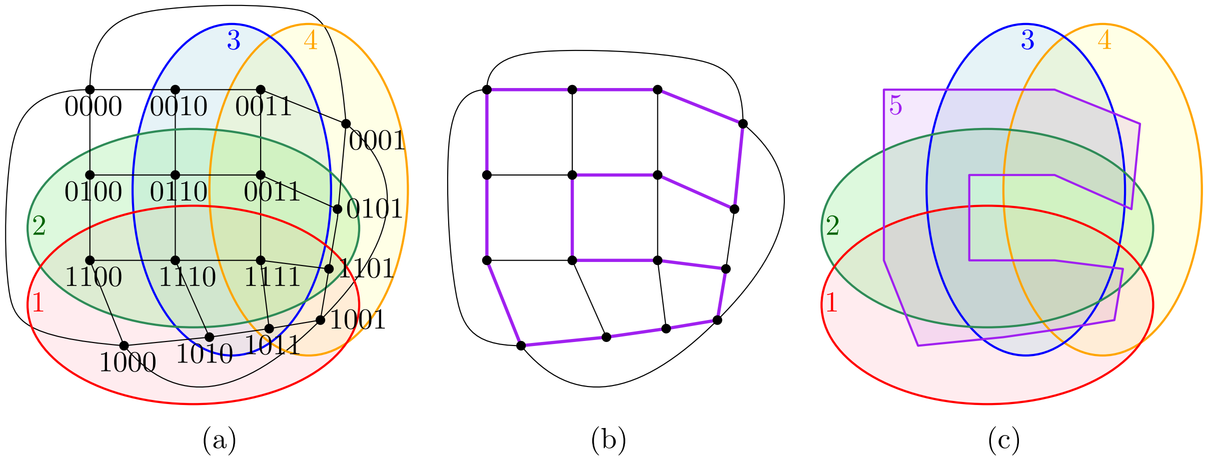

FIGURE 2. (a) A 4-Venn diagram and its dual graph, a 4-Venn quadrangulation; (b) a Hamilton cycle in the quadrangulation; (c) the 5-Venn diagram obtained from (a) by adding the curve corresponding to the Hamilton cycle in (b).

1.2. **Translation to a Hamilton cycle problem.** Problems on Venn diagrams such as Winkler's conjecture are best discussed by considering the dual graph of the Venn diagram; see Figure 2. For this, we consider the *n-dimensional hypercube* Q_n, or *n-cube* for short, the graph formed by all bitstrings of length n, with an edge between any two bitstrings that differ in a single bit. The dual graph $Q(D)$ of a simple n-Venn diagram D satisfies the following properties:

① It is a spanning subgraph of Q_n, i.e., all 2^n vertices are present. Specifically, the vertex in $Q(D)$ corresponding to a region of the diagram D is the characteristic vector of the inside/outside relation with respect to each curve, where 0 and 1 represent outside and inside, respectively.[1]

② It is a plane quadrangulation, i.e., it is drawn in the plane without edge crossings, and every face is a 4-cycle.

③ For every position $i \in \{1, \ldots, n\}$ and every bit $b \in \{0, 1\}$, the subgraph of $Q(D)$ induced by all vertices x with $x_i = b$ is connected.

Conversely, the dual of any subgraph of Q_n satisfying these properties is an n-Venn diagram. We refer to subgraphs of Q_n satisfying ①–③, which are the duals of n-Venn diagrams, as *n-Venn quadrangulations*.

Extending the diagram D by a single curve means that the new curve has to enter and leave every region of D exactly once, splitting it into two, one of them inside and the other one outside the newly added curve. This corresponds to traversing a Hamilton cycle in the dual graph $Q(D)$, i.e., a cycle in the graph that visits every vertex exactly once; see Figure 2 (b)+(c).

Conjecture 1 can thus be restated equivalently as follows: For $n \geq 2$, every n-Venn quadrangulation admits a Hamilton cycle.

1.3. **The counterexample.** As Q_n is a bipartite graph, Venn quadrangulations are bipartite, too. A necessary condition for the existence of a Hamilton cycle in a bipartite graph is the existence of a perfect matching. We provide a counterexample to Conjecture 1 by constructing a 7-Venn quadrangulation that has no perfect matching and thus no Hamilton cycle; see Figures 5–7.

[1]When considering the diagram in the plane, then inside and outside of a curve C are the bounded and unbounded region of $\mathbb{R}^2 \setminus C$, respectively, but when considering the diagram on a sphere then inside and outside can be fixed arbitrarily.

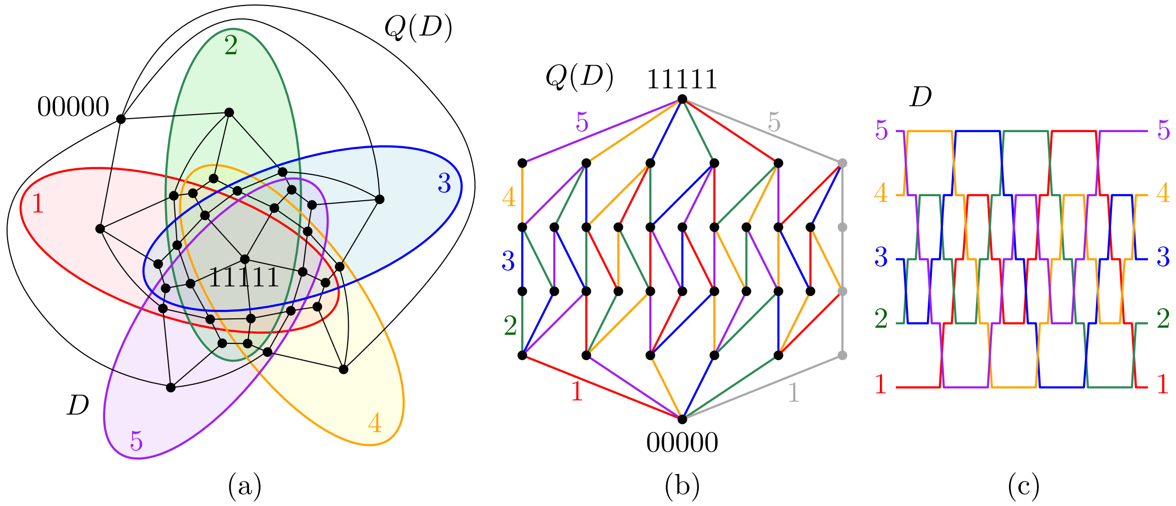

FIGURE 3. (a) A monotone 5-Venn diagram D and its dual 5-Venn quadrangulation $Q(D)$. (b) Drawing of $Q(D)$ as a rhombic strip of Q_5, where each edge is colored according to the bit flipped along this edge. The leftmost vertices are duplicated in gray at the right boundary, to facilitate recognizing the cylindrical embedding. (c) Another drawing of the Venn diagram D as a wire diagram based on (b).

Theorem 2. *There is a 7-Venn quadrangulation that has no perfect matching and hence no Hamilton cycle. Its dual is a simple 7-Venn diagram that cannot be extended to a simple 8-Venn diagram by adding a suitable curve.*

Proof. The 7-Venn quadrangulation of Figure 5 has no perfect matching, because it contains a set of 18 vertices from one partition class (marked red) whose neighborhood (marked blue) has only size 17. This is the easy direction of Hall's theorem. □

2. The construction

The counterexample to Conjecture 1 displayed in Figure 5 was constructed partly by hand, and partly with computer assistance through SAT solvers. In the following, we describe these steps in some detail. Throughout this section, we only consider simple Venn diagrams.

2.1. **Monotone diagrams and rhombic strips.** A *k-region* in an n-Venn diagram is a region that lies inside of exactly k of the n curves (and outside of the remaining $n-k$ curves). A *monotone* Venn diagram D is one in which for every $0 < k < n$, every k-region is adjacent to both a $(k-1)$-region and a $(k+1)$-region. In the dual quadrangulation $Q(D)$, a k-region corresponds to a vertex of Q_n with *weight k*, i.e., the corresponding bitstring has exactly k many 1s and $n-k$ many 0s. We refer to the set of all vertices of Q_n with weight k as the *level k*.

A *rhombic strip* of Q_n is defined as follows; see Figure 3 (b):

(1') It is a spanning subgraph of Q_n, i.e., all 2^n vertices are present.

(2') It is a plane quadrangulation in which vertices with weight k are embedded at vertical coordinate k, edges are drawn as (straight) line segments without crossings between consecutive levels, and every face is a 4-cycle that spans three consecutive levels. The embedding wraps around at the left and right boundary like on a cylinder, or equivalently, on a sphere.

We observe that rhombic strips in Q_n are precisely the duals of monotone (simple) n-Venn diagrams. Indeed, the connectivity condition (3) is implied by observing that for any vertex x

of weight k in a rhombic strip, there is a path of length k to the vertex 0^n along which the weight decreases monotonically, i.e., every vertex along this path has a 0-bit at every position at which x has a 0. Symmetrically, there is a path of length $n-k$ to the vertex 1^n along which the weight increases monotonically, and therefore every vertex along this path has a 1-bit at every position at which x has a 1. Consequently, connectivity is ensured via the vertices 0^n and 1^n at the bottom and top, respectively.

We note that rhombic strips have recently been investigated in [ACF+25, BBM+24] for more general classes of ranked posets (other than Q_n ranked by weights).

2.2. **Neighborhood size violation.** Our aim is to construct a Venn quadrangulation that has no perfect matching (and hence no Hamilton cycle). To this end, we seek a subset U of vertices from one partition class of the graph such that the neighborhood $\Gamma(U)$ is strictly smaller, i.e.,

$$|U| > |\Gamma(U)|. \tag{1}$$

We can find such subgraphs in rhombic strips (i.e., in the setting of monotone Venn diagrams) relatively easily. Specifically, for a sequence $b = (b_s, b_{s+1}, \ldots, b_t)$ of integers with $b_s = b_t = 1$ and $b_k \geq 3$ for all $k = s+1, \ldots, t-1$, a *b-strip* of Q_n is a subgraph of Q_n for some $n \geq t$ with exactly b_k vertices of weight k for all $k = s, \ldots, t$, embedded in the plane such that all weight k vertices have the vertical coordinate k, edges are drawn as line segments without crossings, every face except the outer face is a 4-cycle that spans three consecutive levels, and there are edges between any two leftmost vertices of consecutive levels and any two rightmost vertices of consecutive levels; see Figure 4.

As the set U in the b-strip, we may take all but the two boundary vertices on every second level (marked red in Figure 4), giving

$$|U| = \sum_{i\geq 1:\ s+2i<t} (b_{s+2i} - 2).$$

Then the neighborhood of U is given by all other levels between and around vertices from U (marked blue in the figure), i.e., we obtain

$$|\Gamma(U)| = \sum_{i\geq 0:\ s+2i+1\leq t} b_{s+2i+1}.$$

The smallest instances of b-strips that we found and that satisfy (1) are for $n = 6$, $s = 0$ and $t = 6$; see Figure 4 (b)–(e). We restricted this search to sequences b that are symmetric, i.e., of the form $b = (1, \alpha, \beta, \gamma, \beta, \alpha, 1)$. In this case, we have $|U| = 2(\beta - 2)$ and $|\Gamma(U)| = 2\alpha + \gamma$ and the inequality (1) becomes

$$2\beta - 4 > 2\alpha + \gamma. \tag{2}$$

The examples in Figure 4 (b)–(e) satisfy this inequality and they all have $\alpha = 3$. In fact, the b-strip in part (e) of the figure is part of the counterexample shown in Figure 5.

2.3. **SAT solving.** Our next goal is to extend one of the b-strips in Q_6 satisfying (1), such as the ones from Figure 4 (b)–(e), to a rhombic strip of Q_n or more generally, to an n-Venn quadrangulation for some $n \geq 6$ (ideally $n = 6$). This entails embedding the remaining vertices of Q_n around the prescribed b-strip such that conditions (1')–(2') or (1)–(3), respectively, are satisfied. This task quickly becomes infeasible by hand, which is why we used the SAT solvers Glucose [AS18] and Kissat [BFF+24].

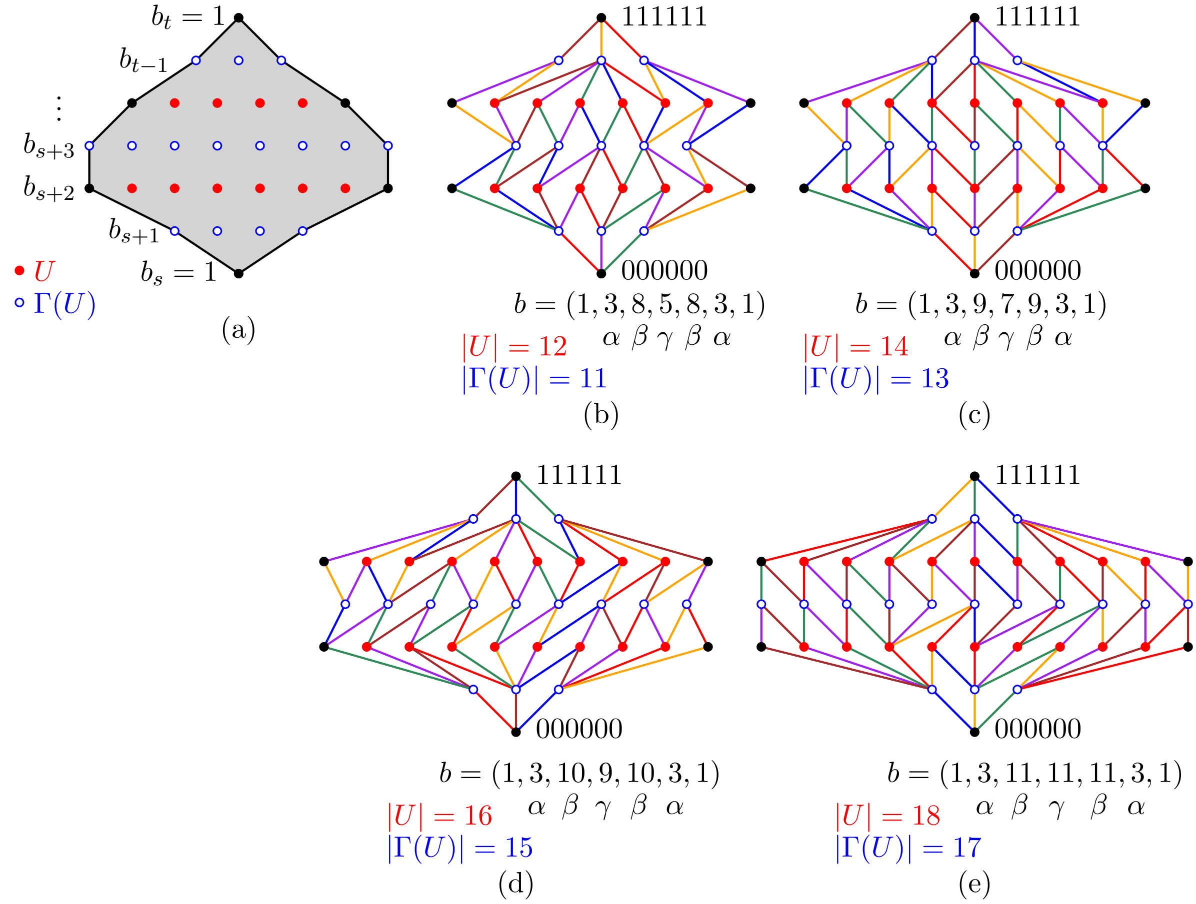

FIGURE 4. (a) Schematic illustration of a b-strip, where the gray area indicates edges that form 4-cycles as faces; (b)–(e) are concrete instances of b-strips of Q_6 that satisfy condition (1), more specifically, inequality (2). Edges of the same color have the same bit flipped along them.

2.3.1. *Encoding rhombic strips.* The conditions (1') and (2') of a rhombic strip of Q_n can be encoded straightforwardly as a SAT formula Φ_n as follows. For all $k = 0, \dots, n$, we label all vertices in level k of Q_n arbitrarily with integers $1, \dots, \binom{n}{k}$. We use variables $e_{k,i,j}$ to indicate that the edge of Q_n between vertex i at level k and vertex j at level $k+1$ is included in the rhombic strip. For non-edges we prescribe the value $e_{k,i,j} = 0$ by adding the unit clause $[\neg e_{k,i,j}]$.

To embed all level k vertices in the rhombic strip, we choose w.l.o.g. the points (j, k) for $j = 0, \dots, \binom{n}{k} - 1$. We use variables $f_{k,i,i'}$ to indicate that the ith vertex of Q_n in level k is embedded at the point (i', k). Suitable clauses ensure that exactly one vertex is embedded at each point. Furthermore, we use variables $\ell_{k,i,j}$ to indicate that the rhombic strip has a line segment between the points (i, k) and $(j, k+1)$. The coupling between edges of the graph and line segments in the embedding is achieved by clauses of the form $[\neg f_{k,i,j}, \neg f_{k+1,i',j'}, \neg \ell_{k,j,j'}, e_{k,i,i'}]$, which expresses that if two vertices are embedded to the jth and j'th point at levels k and $k+1$, respectively, and an edge is drawn between these two points, then the graph has to have an edge between the embedded vertices.

To capture the cylindrical embedding, we add an artificial last point $(\binom{n}{k}, k)$ at the right side of each level (except for the bottom and top vertex at levels $k = 0$ and $k = n$, respectively), and we enforce that the vertex embedded onto this artificial point is the same as the one embedded onto the first point $(0,k)$. Furthermore, w.l.o.g., we enforce the line segments between any two consecutive leftmost points, and any two consecutive rightmost (artificial) points, by adding the unit clauses $[\ell_{(k,0,0)}]$ and $[\ell_{(k,\binom{n}{k},\binom{n}{k+1})}]$. This simplifies the non-crossing constraints of the edges, which can then be captured by clauses of the form $[\neg\ell_{k,i,j},\neg\ell_{k,i',j'}]$ for all $i'<i$ and $j'>j$.

It remains to add clauses ensuring that each face is a 4-cycle spanning three consecutive levels. For this, we introduce variables $\widehat{c}_{k,i,j}$ and $\check{c}_{k,i,j'}$ to indicate that the 4-cycle that contains the points (i,k) and $(i+1,k)$ has $(j,k+1)$ as its top point in level $k+1$ and $(j',k-1)$ as its bottom point in level $k-1$, respectively. These variables are coupled to the edge variables via the clauses $[\neg\widehat{c}_{k,i,j},e_{k,i,j}]$, $[\neg\widehat{c}_{k,i,j},e_{k,i+1,j}]$, $[\neg\check{c}_{k,i,j'},e_{k-1,j',i}]$, $[\neg\check{c}_{k,i,j'},e_{k-1,j',i+1}]$. Furthermore, we add suitable clauses to ensure that for fixed k and i at least one of the variables $\widehat{c}_{k,i,j}$ and $\check{c}_{k,i,j'}$ is satisfied. In fact, combined with the aforementioned non-crossing constraints, exactly one of them will be satisfied.

By construction, solutions of this formula Φ_n are in one-to-one correspondence to rhombic strips of Q_n. Furthermore, we can add unit clauses to prescribe all edges of a b-strip, and test whether the b-strip extends to a full rhombic strip. In fact, given the vector $b=(b_s,b_{s+1},\ldots,b_t)$, it is enough to prescribe the line segments $[\ell_{k,b_s-1,b_{s+1}-1}]$ for all $k=s,s+1,\ldots,t-1$ and thus test whether there is a b-strip that extends to a rhombic strip. Note that for the counterexample we do not really care about the edges inside the b-strip, but only about the numerical values of the entries of the vector b. By disabling the constraints outside of the b-strip, we may even search for b-strips alone, without a surrounding rhombic strip.

Table 1 shows the size of the formula Φ_n for $n=3,4,5,6,7$. Up to $n=6$ the solvers report solutions or unsatisfiability within seconds or minutes, whereas for $n=7$, the formula is apparently too big to yield results within reasonable time. Unfortunately, we did not obtain a full rhombic strip of Q_6 nor Q_7 that would extend one of the b-strips shown in Figure 4 (b)–(e), but we did find many more b-strips than the ones in the figure.

2.3.2. *Encoding Venn quadrangulations.* The next goal is to allow more flexibility when extending a b-strip, by dropping the requirement that the resulting Venn diagram is monotone. Unfortunately, it seems highly nontrivial to encode the conditions ①–③ of a general Venn quadrangulation as a SAT formula. The main reason is that we cannot easily assume certain points in the plane where the vertices will be embedded. This is in contrast to rhombic strips, where we could assume w.l.o.g. certain integer coordinates and there is a one-to-one mapping between all 2^n vertices and points.

We thus construct a much smaller formula Ψ_n that captures only some necessary conditions for a subgraph H of Q_n to be an n-Venn quadrangulation. This will lead to false positives, i.e., some solutions of Ψ_n correspond to a subgraph H of Q_n that is not an n-Venn quadrangulation, which can only be checked after the SAT solver returns a solution. By running the solvers many times with different random initializations and on randomized versions of Ψ_n (obtained by permuting the variable names and the clauses), one hopes to sometimes obtain a solution H that actually is an n-Venn quadrangulation. On the other hand, in the case of unsatisfiability certified by the solver, we can be sure that no such Venn diagram exists, in particular no monotone one.

Specifically, our formula Ψ_n has a variable x_e for every edge e of Q_n to indicate whether the edge e is present in the subgraph H. Furthermore, for every 4-cycle f of Q_n we have a variable y_f to indicate whether this 4-cycle is present as a face in H. The coupling between a 4-cycle f and the four edges e_1,e_2,e_3,e_4 contained in f is provided via the four clauses $[\neg y_f,x_{e_i}]$ for $i=1,2,3,4$. Every 4-cycle f comes with two extra variables $\overrightarrow{y_f}$ and $\overleftarrow{y_f}$ that indicate whether it is embedded as a face in clockwise or counterclockwise order, respectively. This is determined by

Table 1. Sizes of our SAT formulas (without partially prescribing a solution).

	Formula Φ_n		Formula Ψ_n	
	#variables	#clauses	#variables	#clauses
$n=3$	95	341	30	153
$n=4$	344	2.686	104	756
$n=5$	1.217	25.819	320	3.165
$n=6$	4.372	289.263	912	12.562
$n=7$	16.039	3.572.925	2.464	55.139
$n=8$			6.400	304.696

the direction in which we see the sequence $00 \to 01 \to 11 \to 10$ on the two variable bit positions of the four vertices along the cycle f. The clauses $[\neg y_f, \overrightarrow{y_f}, \overleftarrow{y_f}]$, $[\neg y_f, \neg\overrightarrow{y_f}, \neg\overleftarrow{y_f}]$ ensure that if a 4-cycle is present as a face, then exactly one of its two possible orientations is set.

As a necessary condition for ②, we add clauses to enforce that if an edge e is present in H, then exactly two of the $n-1$ many 4-cycles $f_1, \ldots, f_{n-1}$ containing e have to be present in H, and their two possible orientations (clockwise or counterclockwise) have to be consistent. Another necessary condition for planarity is to add clauses to enforce that if edges $e_1, \ldots, e_t$ are incident to the same vertex v and all present in H, then for every splitting of the set $\{e_1, \ldots, e_t\}$ into two subsets, at least one 4-cycle f containing one edge from each of the subsets must be present in H.

As a necessary condition for ③, we add clauses to enforce connectivity within subcubes, i.e., for very position $i \in \{1, \ldots, n\}$ and every bit $b \in \{0, 1\}$, we consider the $(n-1)$-cube in Q_n induced by all vertices x with $x_i = b$, and for each of its subcubes, we add a clause enforcing that at least one of the edges leading out of this subcube is present in H. This in particular enforces that at least one edge goes out of each vertex, which implies that all vertices will be present in H, so condition ① is satisfied automatically. Another necessary condition for connectivity is to forbid separating 4-cycles, by adding the clause $[\neg x_{e_1}, \neg x_{e_2}, \neg x_{e_3}, \neg x_{e_4}, y_f]$ for every 4-cycle f and the four edges e_1, e_2, e_3, e_4 contained in f.

Table 1 shows that the formula Ψ_n is considerably smaller than Φ_n, and so it is amenable to SAT solvers in reasonable time even for $n = 7$ and $n = 8$. As mentioned before, the disadvantage is that some of the computed subgraphs H of Q_n violate condition ② or ③. The non-planarity issue becomes less pronounced if a large b-strip is prescribed, because then at least the prescribed part of H is guaranteed to be planar. After running several thousands of randomized instances of Ψ_7 with the b-strip from Figure 4 (e) prescribed (with an extra 0-bit appended), one of the SAT solvers found the Venn quadrangulation of Figure 5, leading to a moment filled with surprise and happiness.

3. Open questions

An obvious question, of course, is whether our counterexample is the smallest one. In other words, are all simple 6-Venn diagrams extendable to a simple 7-Venn diagram? One possible approach to answering this question is to exhaustively generate all simple 6-Venn diagrams, which are currently unknown.

Another natural question is whether Winkler's conjecture holds for monotone Venn diagrams. Equivalently, does every rhombic strip have a Hamilton cycle? We believe that the answer is 'no' and aim to find a counterexample, possibly by extending one of our b-strips from Figure 4 to a rhombic strip in Q_7 or Q_8.

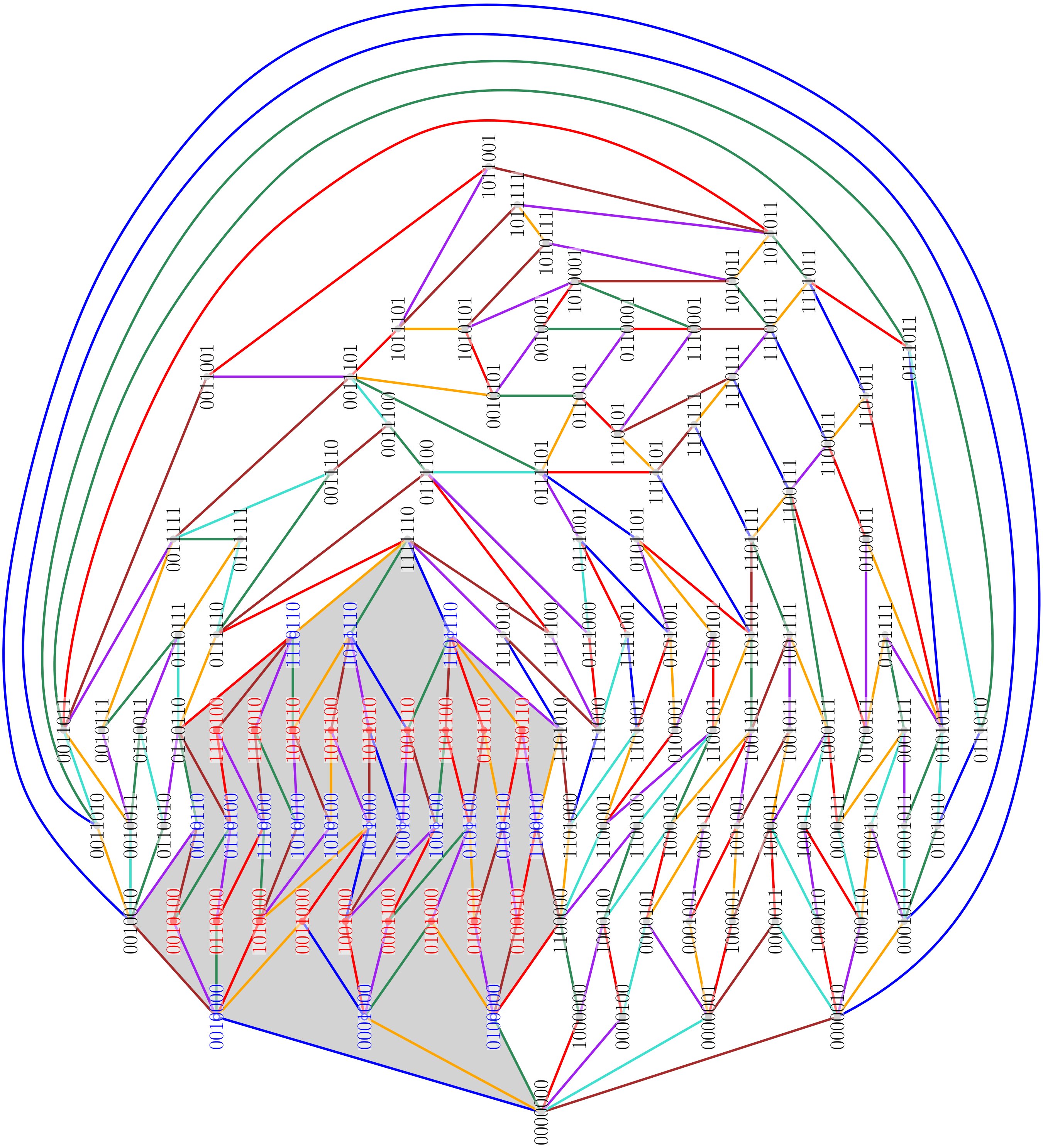

FIGURE 5. A 7-Venn quadrangulation that has no perfect matching and hence no Hamilton cycle. The b-strip from Figure 4 (e) (with an extra 0-bit appended) is shaded gray. A subset of 18 vertices from one partition classes is marked red, and their 17 neighbors are marked blue. The corresponding dual Venn diagram is depicted in Figures 6 and 7.

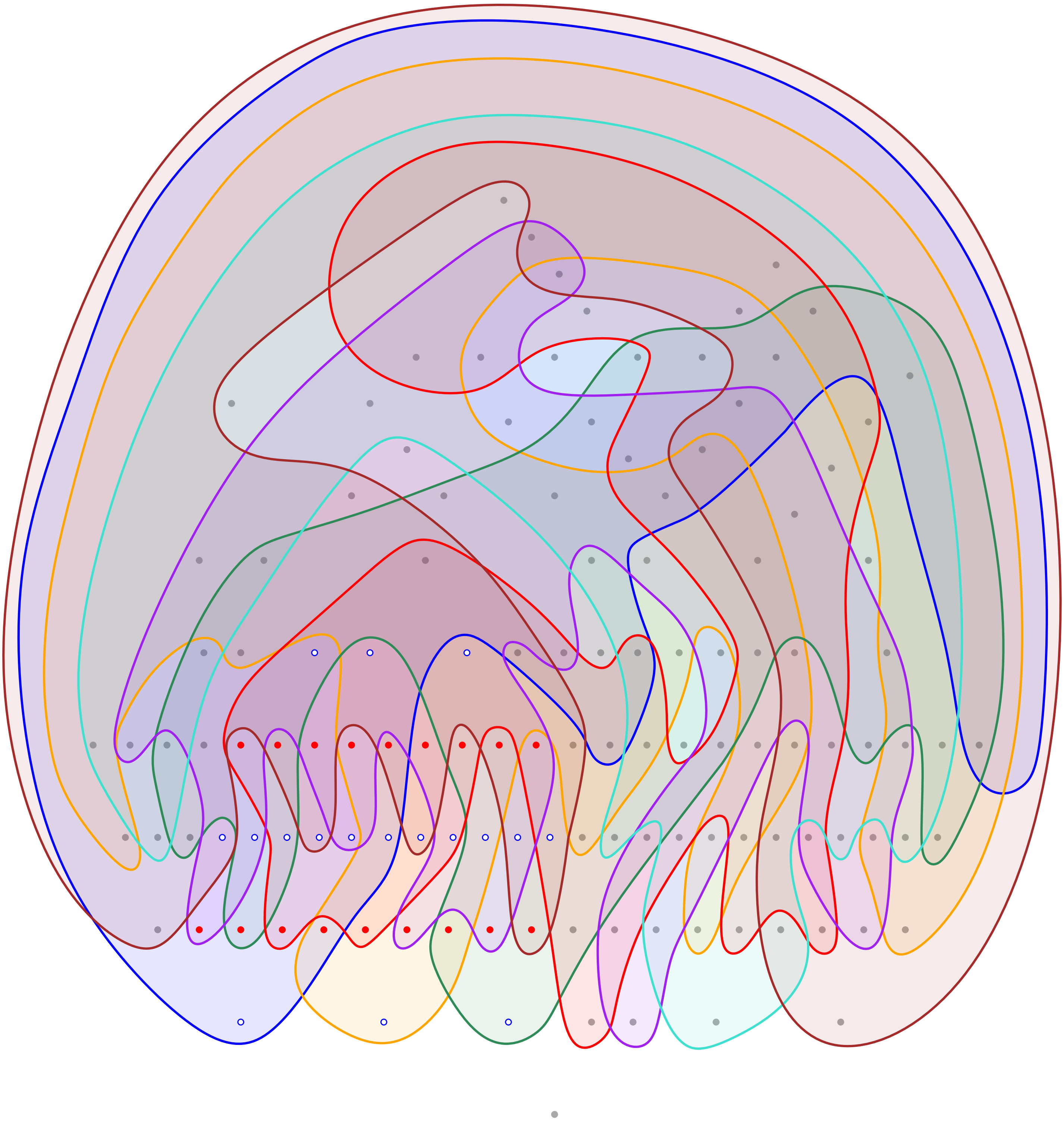

FIGURE 6. Counterexample to Winkler's conjecture, namely a simple 7-Venn diagram that cannot be extended to a simple 8-Venn diagram by adding a suitable curve. It is the dual of the Venn quadrangulation from Figure 5, whose vertices can be seen in the regions.

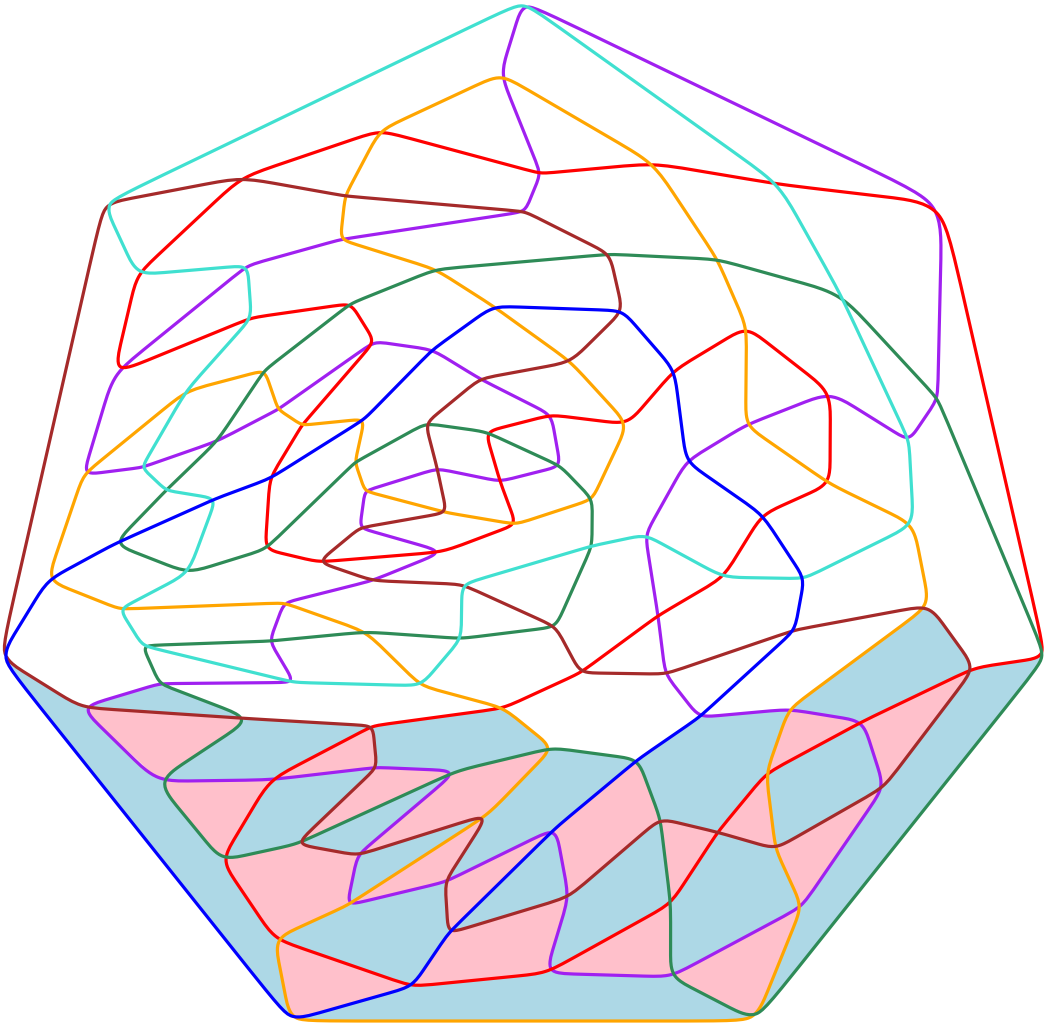

FIGURE 7. Another drawing of the diagram from Figure 6, highlighting 18 red regions that are surrounded by only 17 blue regions, which is a local obstruction for extendability. This drawing was produced using a Sage program [Sch24] developed by Manfred Scheucher for his and Felsner's work [FS21] on pseudocircle arrangements, and we used the program with Scheucher's permission.

In the spirit of the Ruskey-Savage conjecture (cf. [Fin07]) we may ask: Does every perfect matching in a Venn quadrangulation or rhombic strip extend to a Hamilton cycle? Our counterexample to Winkler's conjecture does not have a perfect matching, so the precondition in the question is not satisfied.

In addition to these questions about extendability, there is a large number of other intriguing problems on Venn diagrams, concerning symmetry, convexity, number of crossings, restrictions on the allowed shapes of the curves (triangles, rectangles etc.), to be found in the survey [RW97].

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