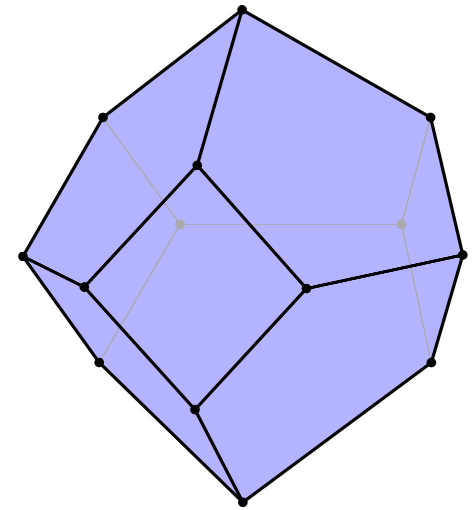
Flips in colorful triangulations

Torsten Mütze
Universität Kassel

joint work with Rohan Acharya (Warwick)
and Francesco Verciani (Kassel)

Introduction



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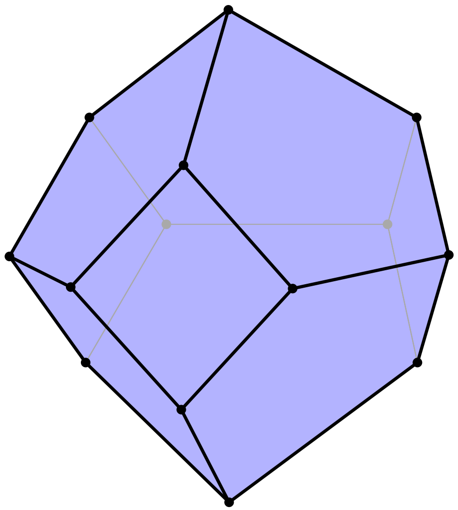
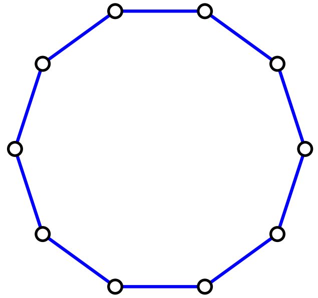


convex N-gon



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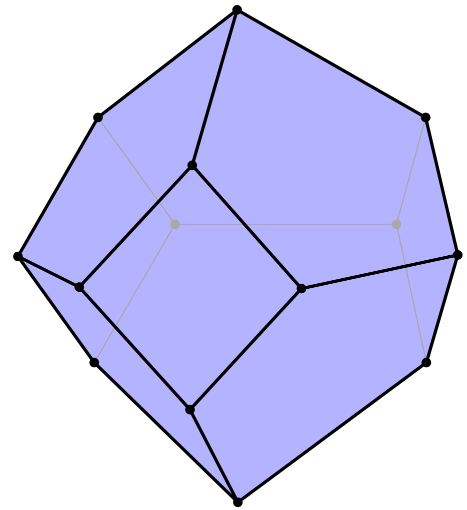

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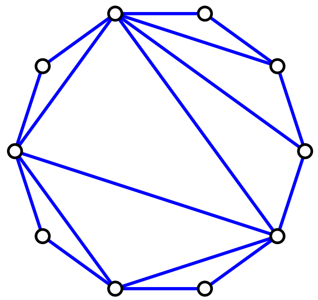

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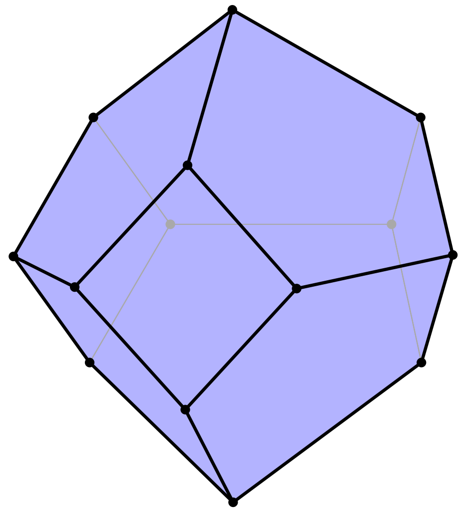

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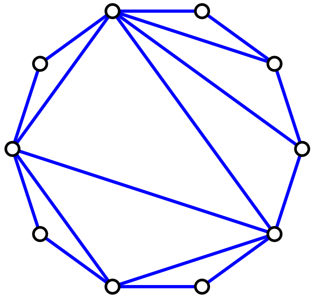
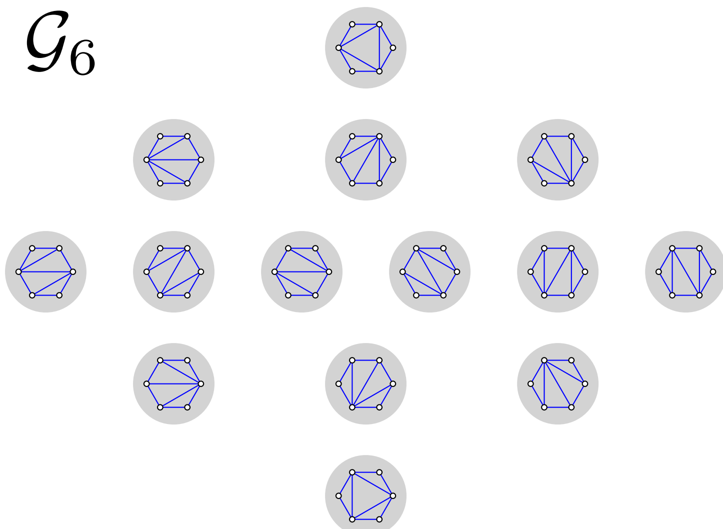


$\mathcal{G}_6$



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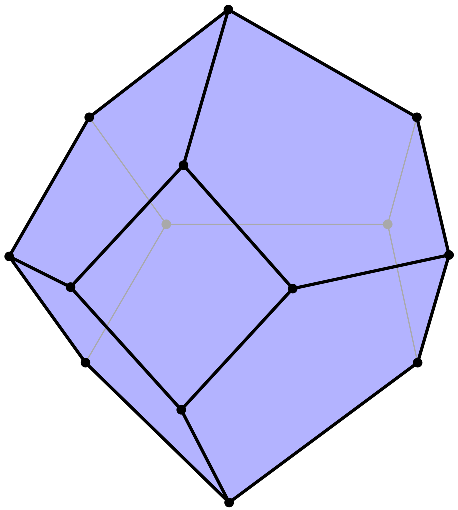

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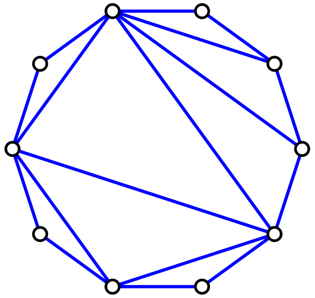

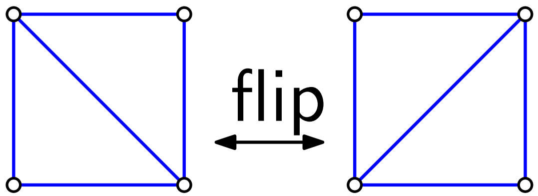

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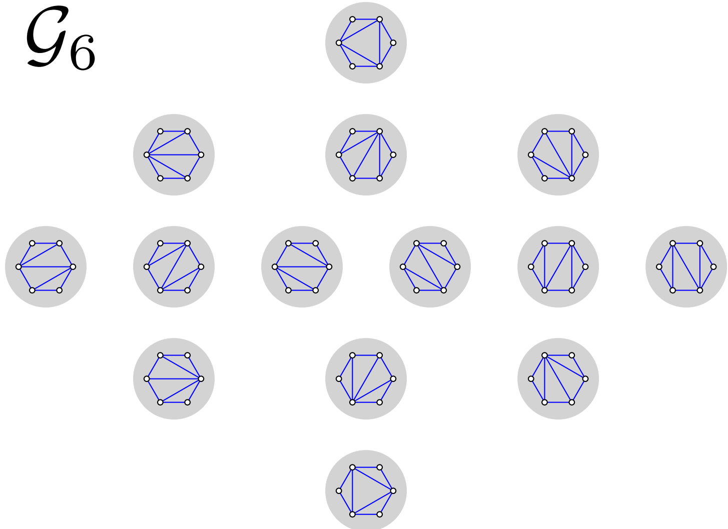

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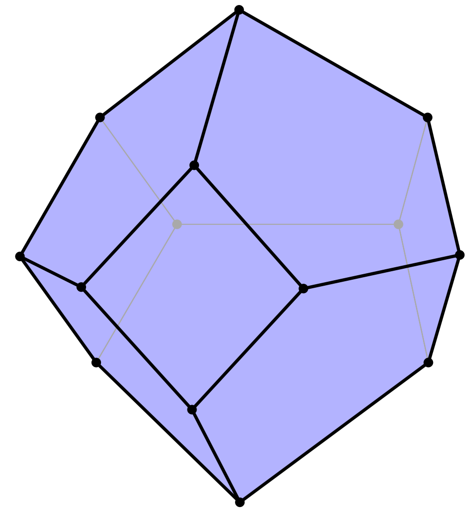

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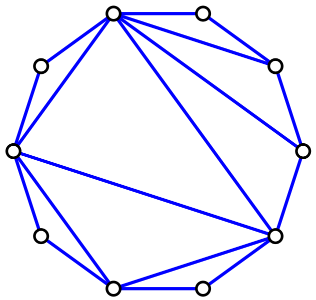

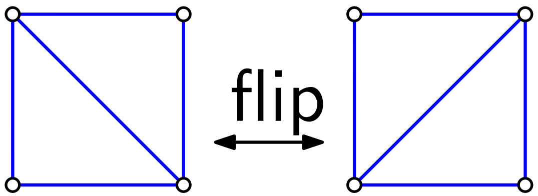

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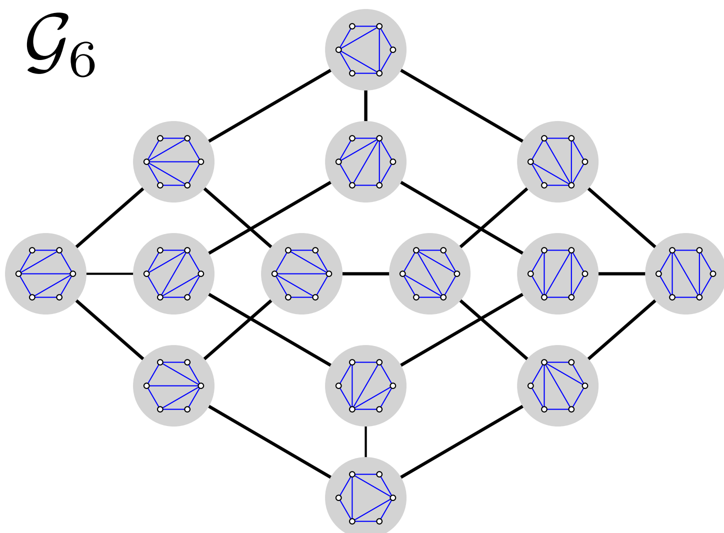

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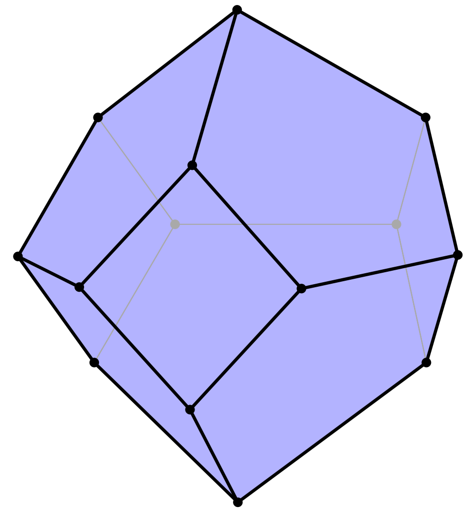

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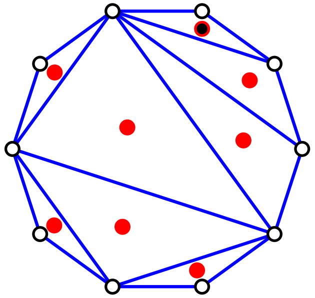

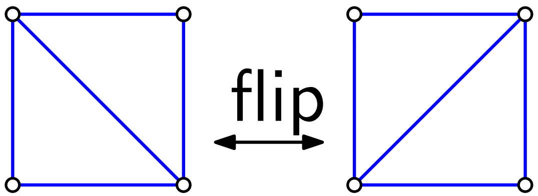

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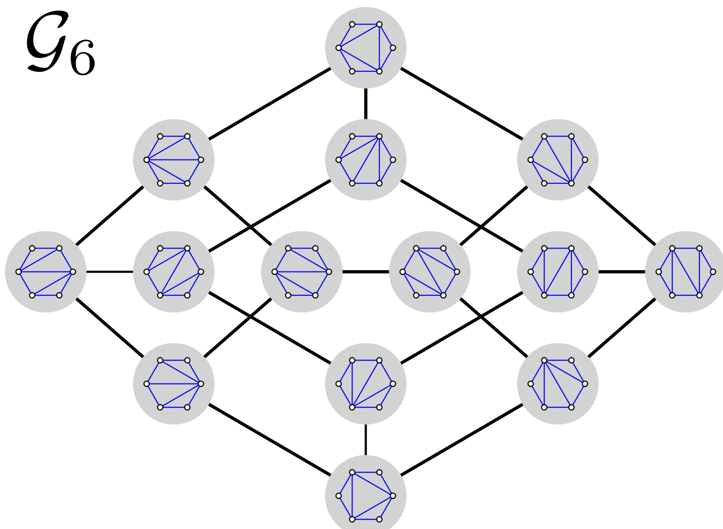

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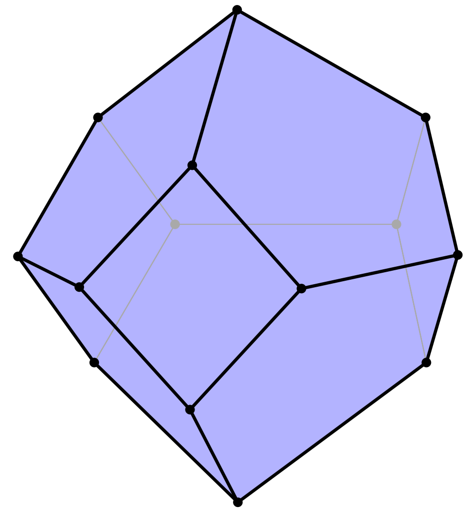

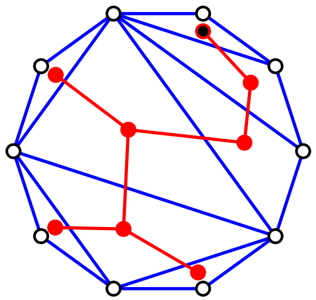

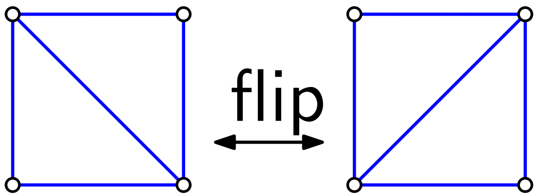

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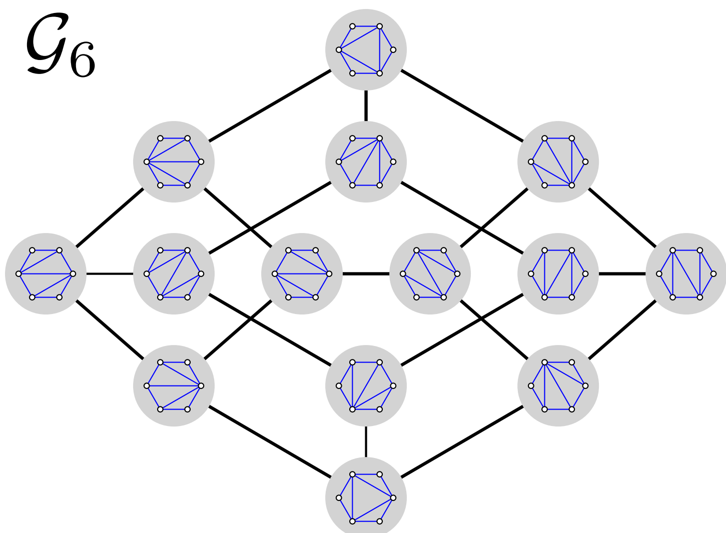

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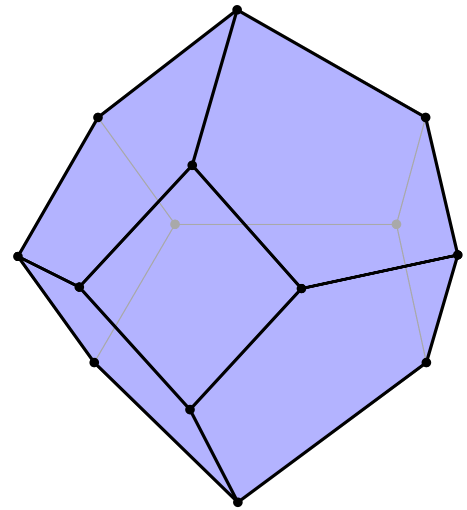

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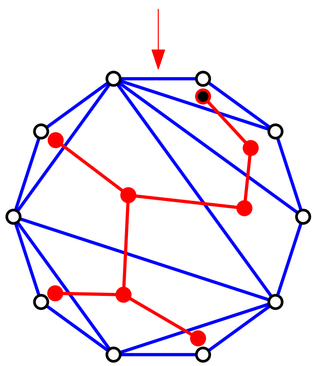

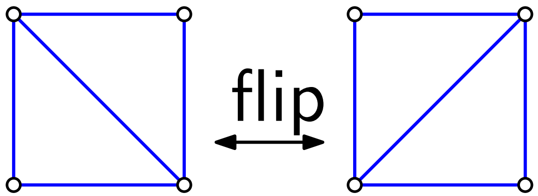

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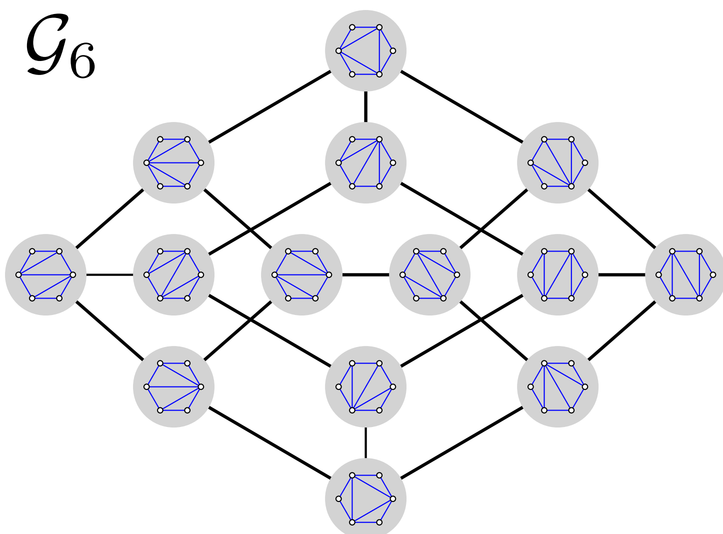

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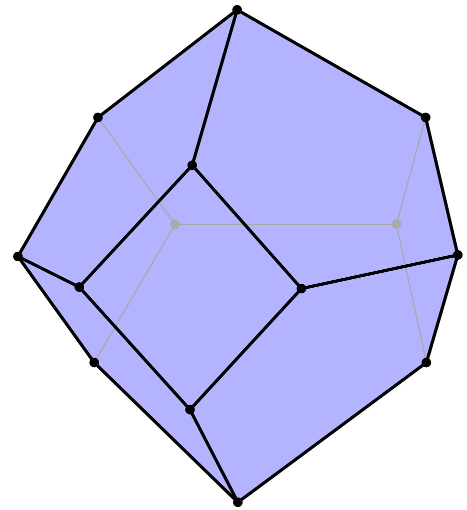

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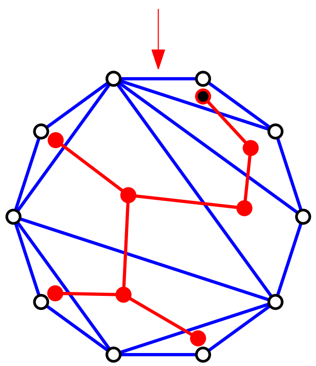
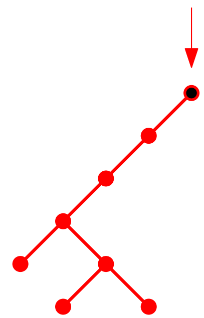
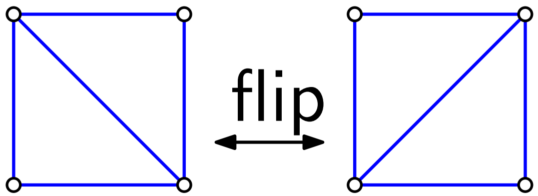




binary tree
$N-2$ vertices



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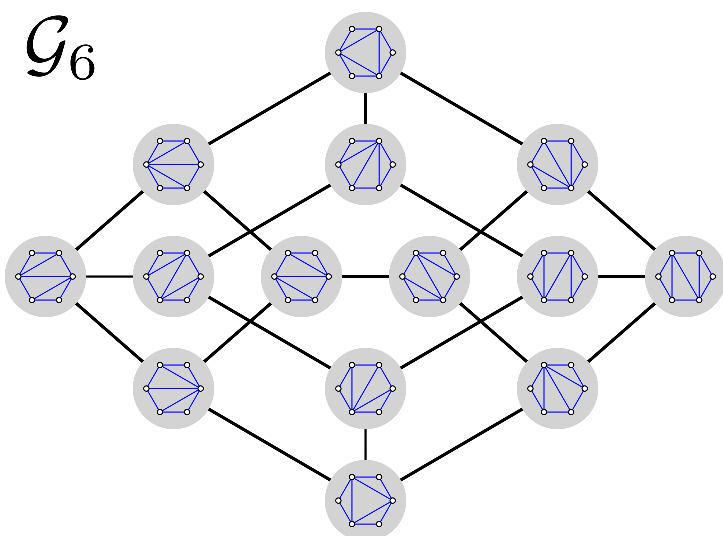

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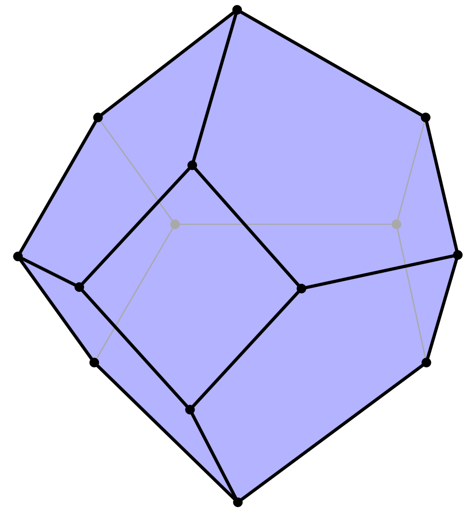

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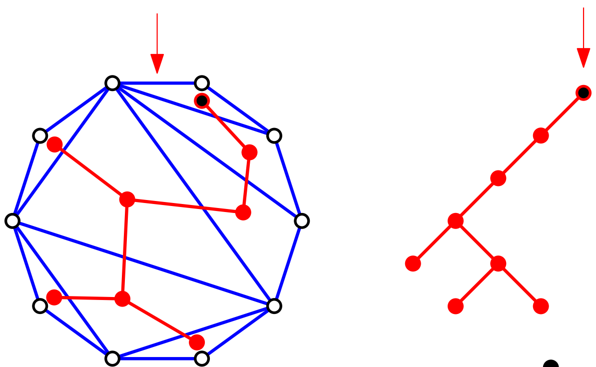

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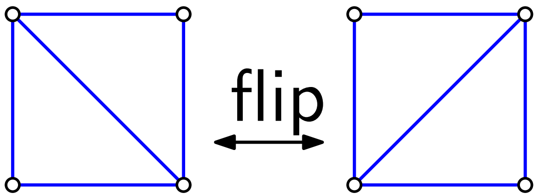

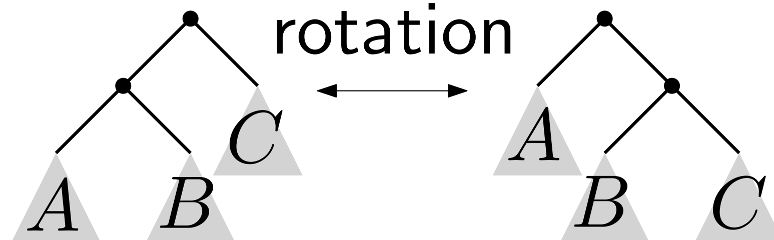

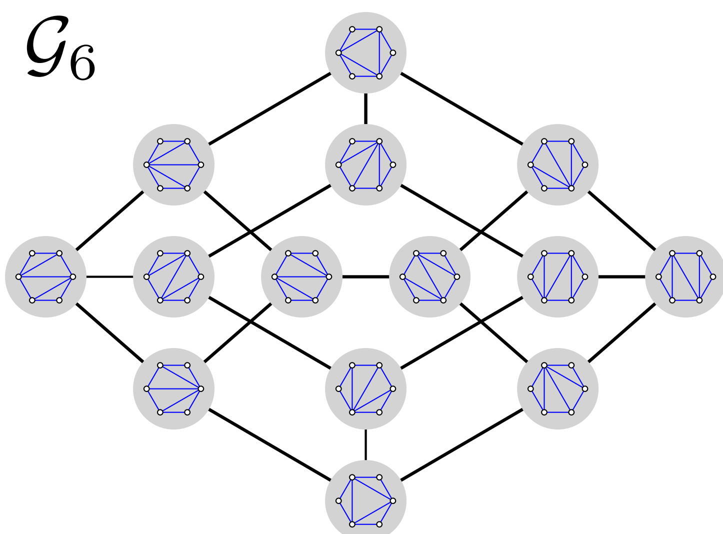

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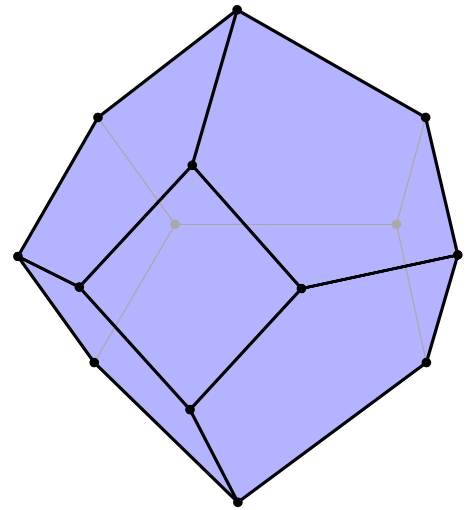

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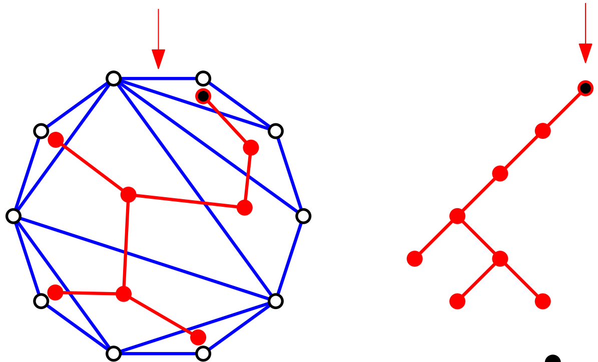

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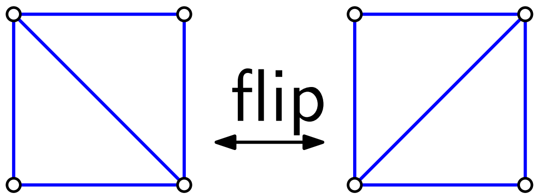

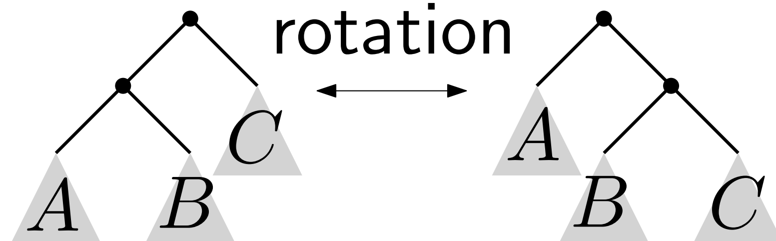

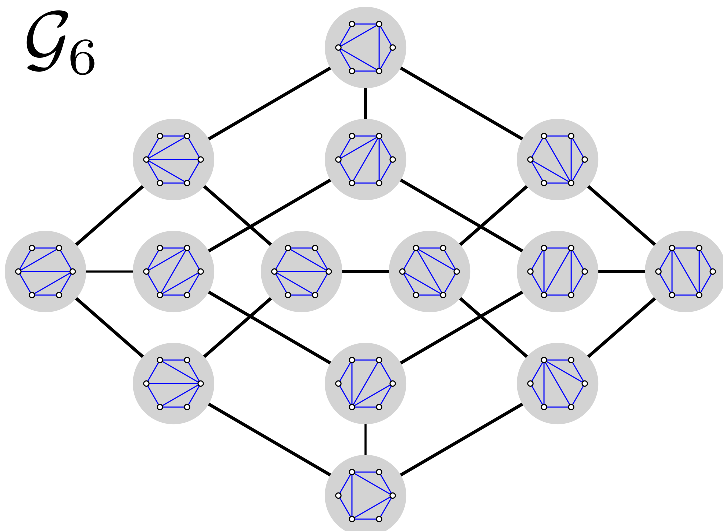

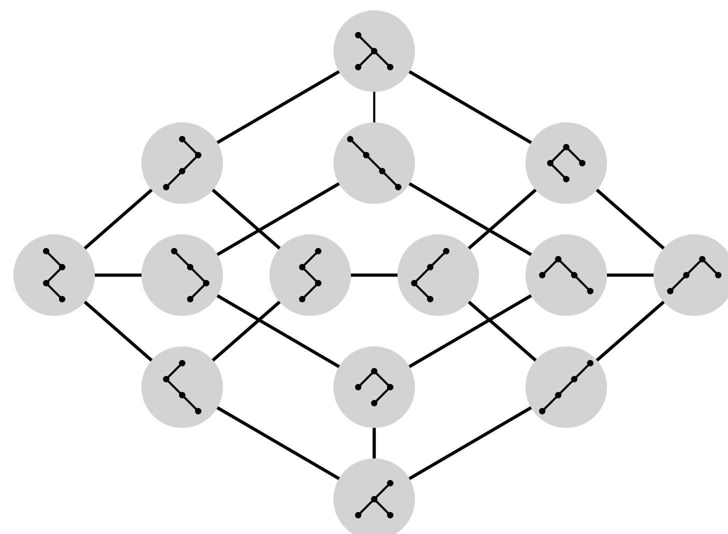

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- Hamilton cycle for $N \geq 5$ [Lucas 87], [Hurtado, Noy 99]

Ramsey-type subgraphs of $\mathcal{G}_N$

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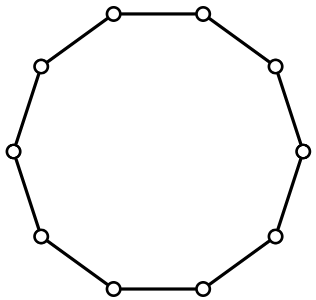

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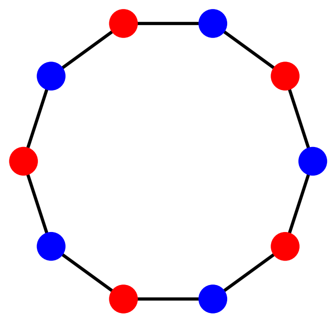

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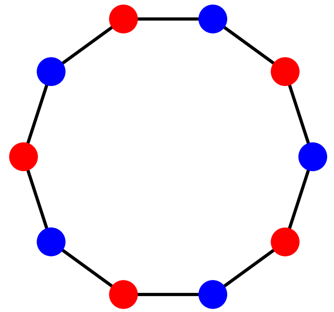
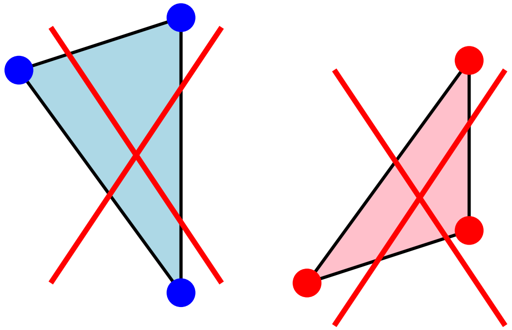


Want: no mono-chromatic triangles



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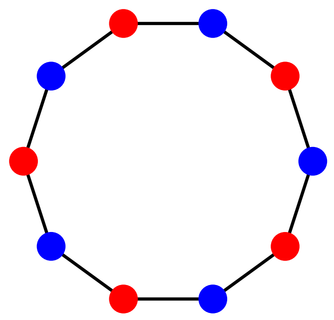

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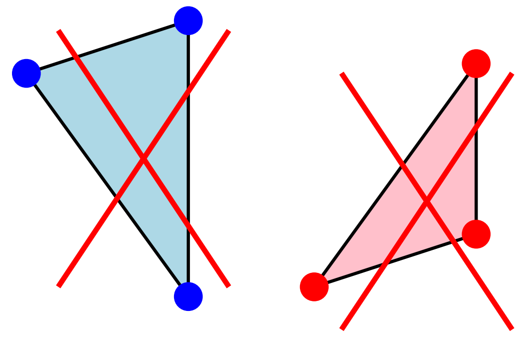

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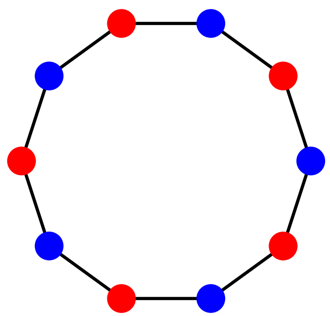

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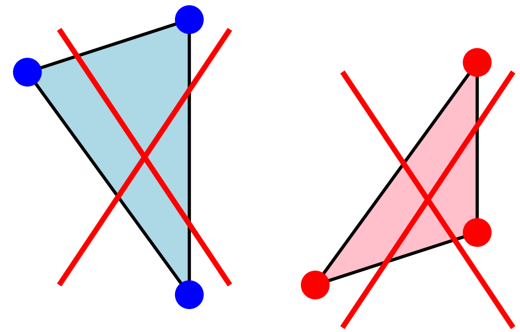

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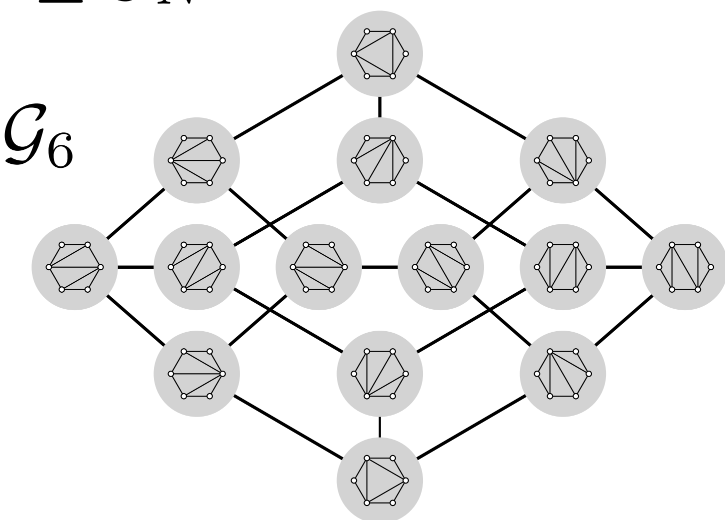

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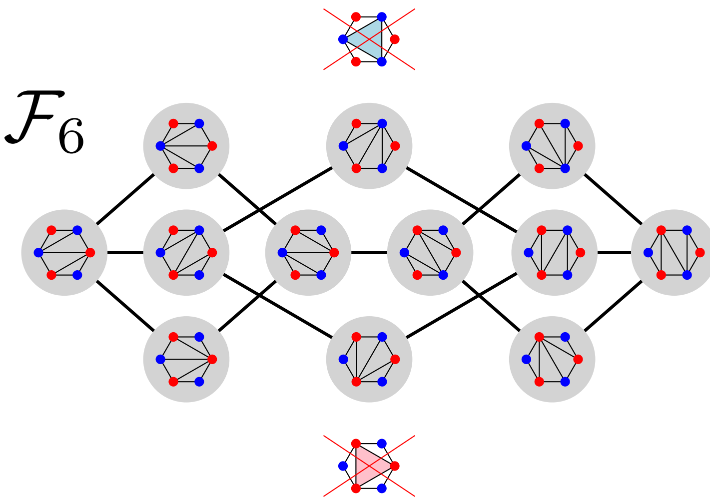

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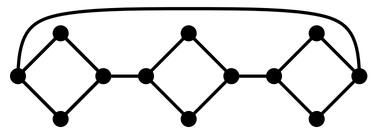
$N = 6$			

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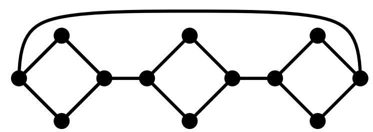

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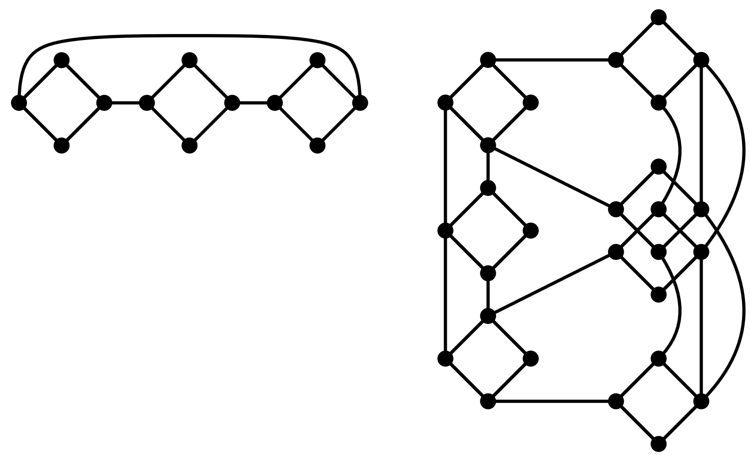

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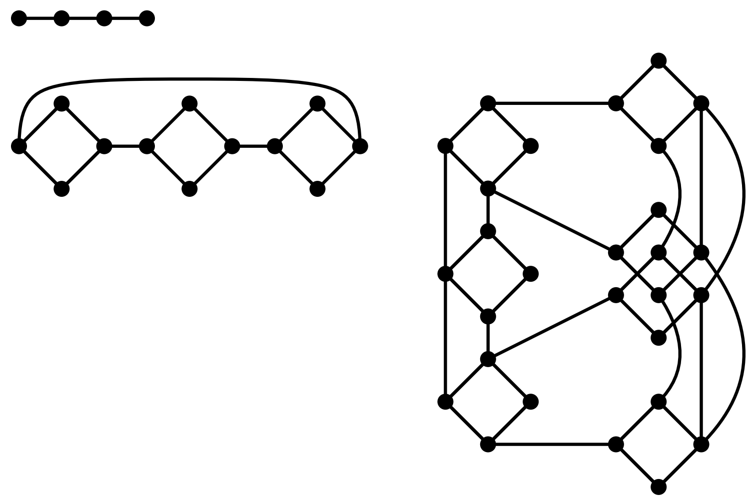

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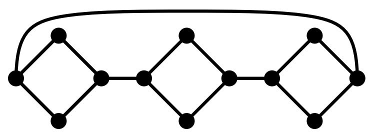
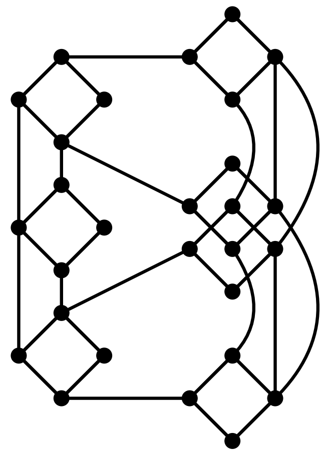
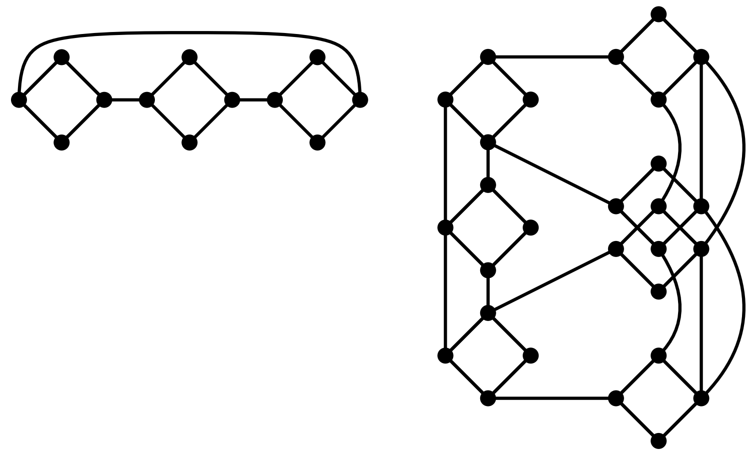



- **Theorem:** For all $N \geq 8$, the graph $\mathcal{F}_N$ has a Hamilton cycle.

Sagan's problem

- Sagan proved that $\mathcal{F}_N$ is connected [Sagan 08]
- He asked: Does $\mathcal{F}_N$ admit a Hamilton path/cycle? [pers. comm.]

	HP	HC
$N = 5$	yes	no
$N = 6$	no	no
$N = 7$	yes	no



- **Theorem:** For all $N \geq 8$, the graph $\mathcal{F}_N$ has a Hamilton cycle.

- can compute Hamilton path in time $\mathcal{O}(1)$ on avg. per node

Structure of $\mathcal{F}_N$

- Assume N even

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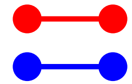
- Distinguish colorful  and monochromatic edges 

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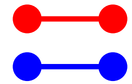
- Distinguish colorful  and monochromatic edges 
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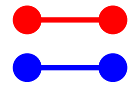
- Distinguish colorful  and monochromatic edges 
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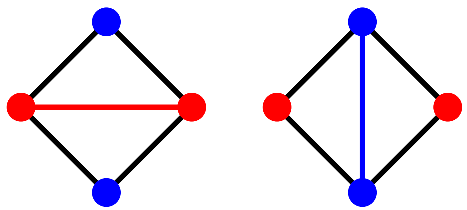
  - every monochr. edges is surrounded by 4 colorful edges



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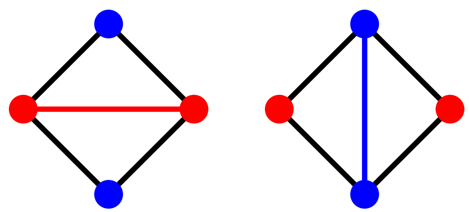

- Removing all monochromatic edges yields a quadrangulation

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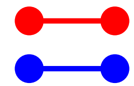
- Distinguish colorful  and monochromatic edges 
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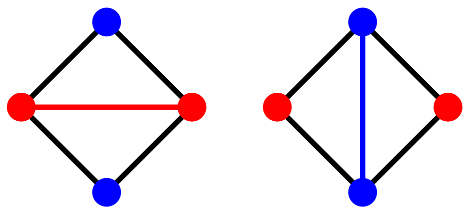

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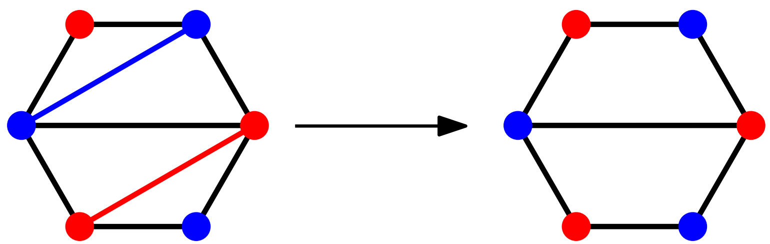
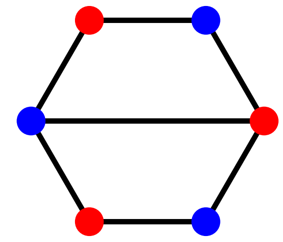


Structure of $\mathcal{F}_N$

- Flips of monochromatic edges are independent and span disjoint hypercubes



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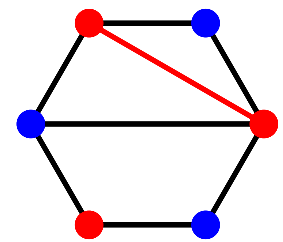

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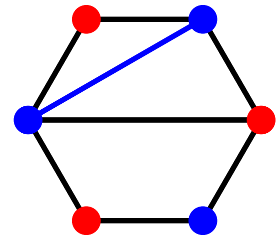

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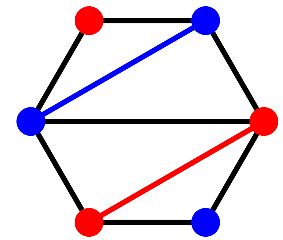

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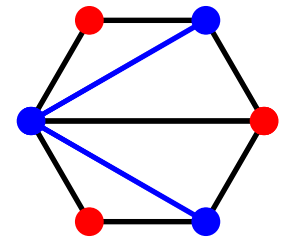

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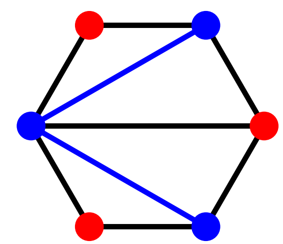

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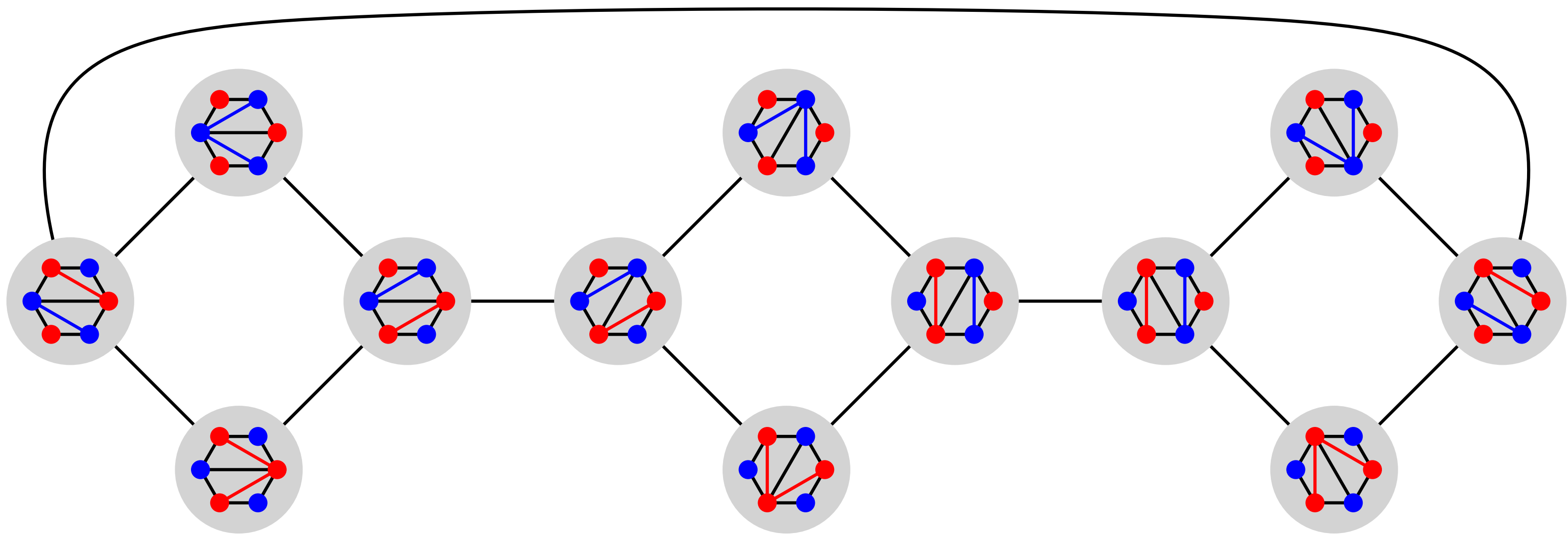
$\mathcal{F}_6$



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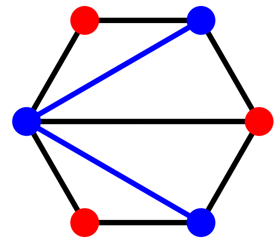

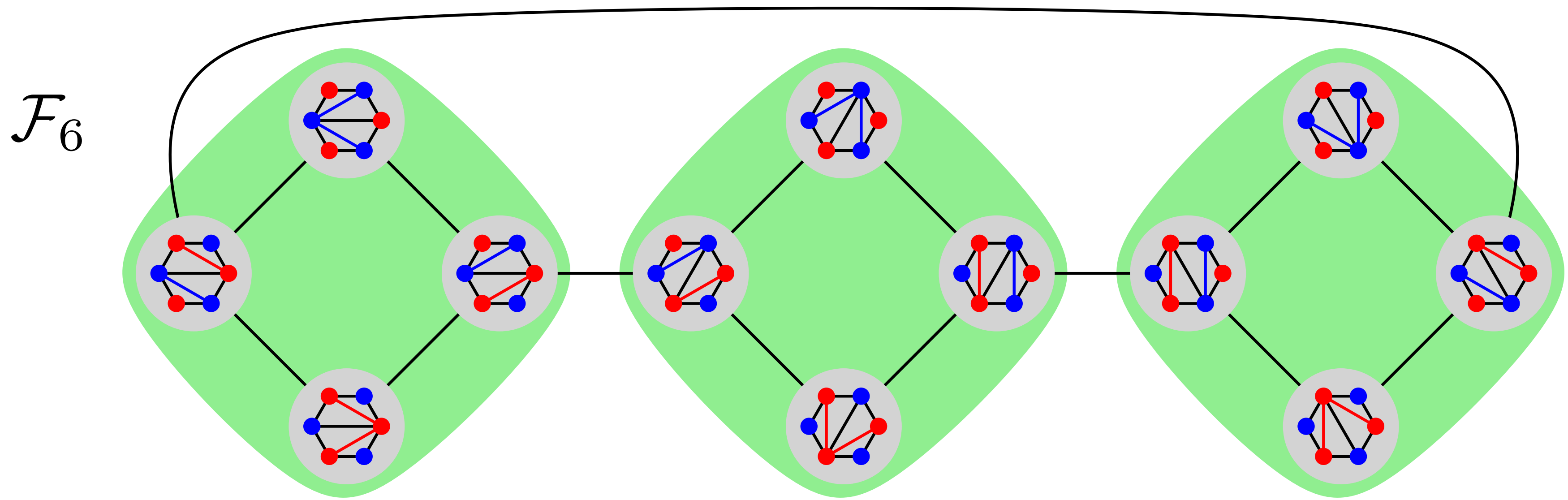

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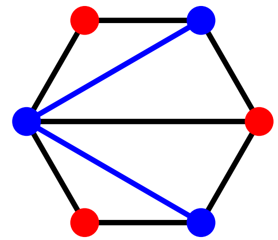

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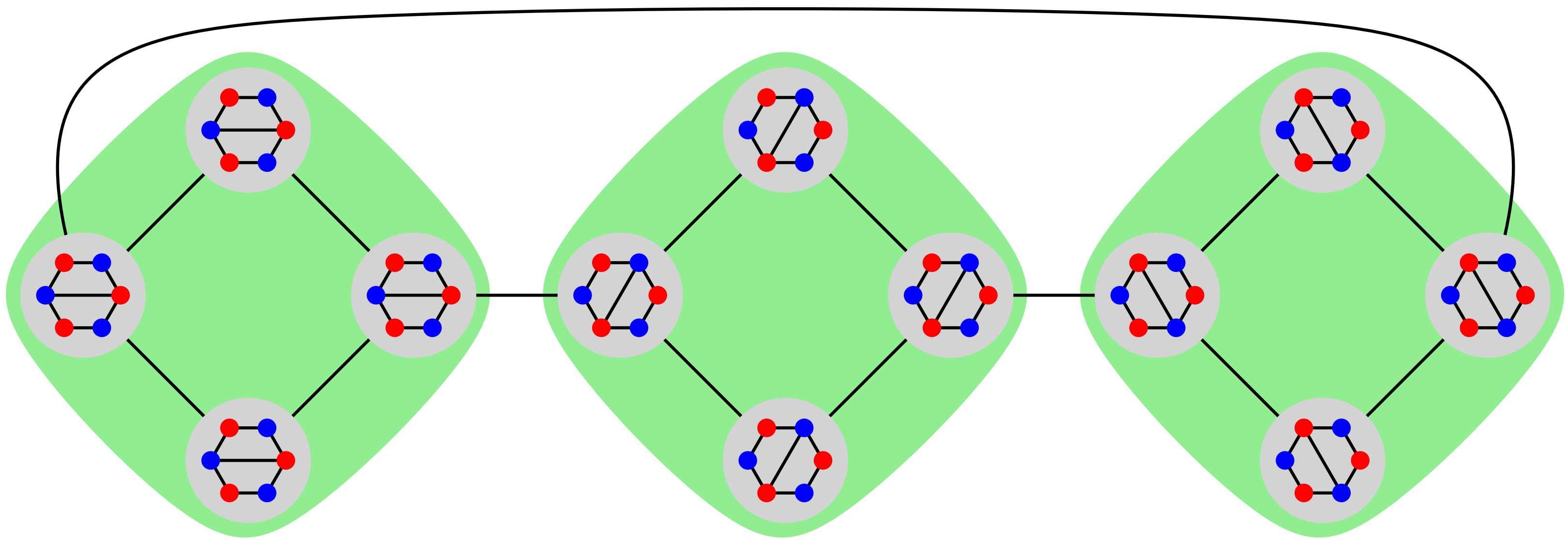

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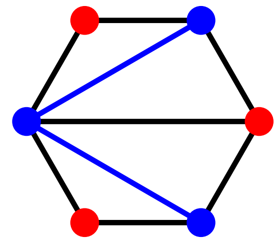

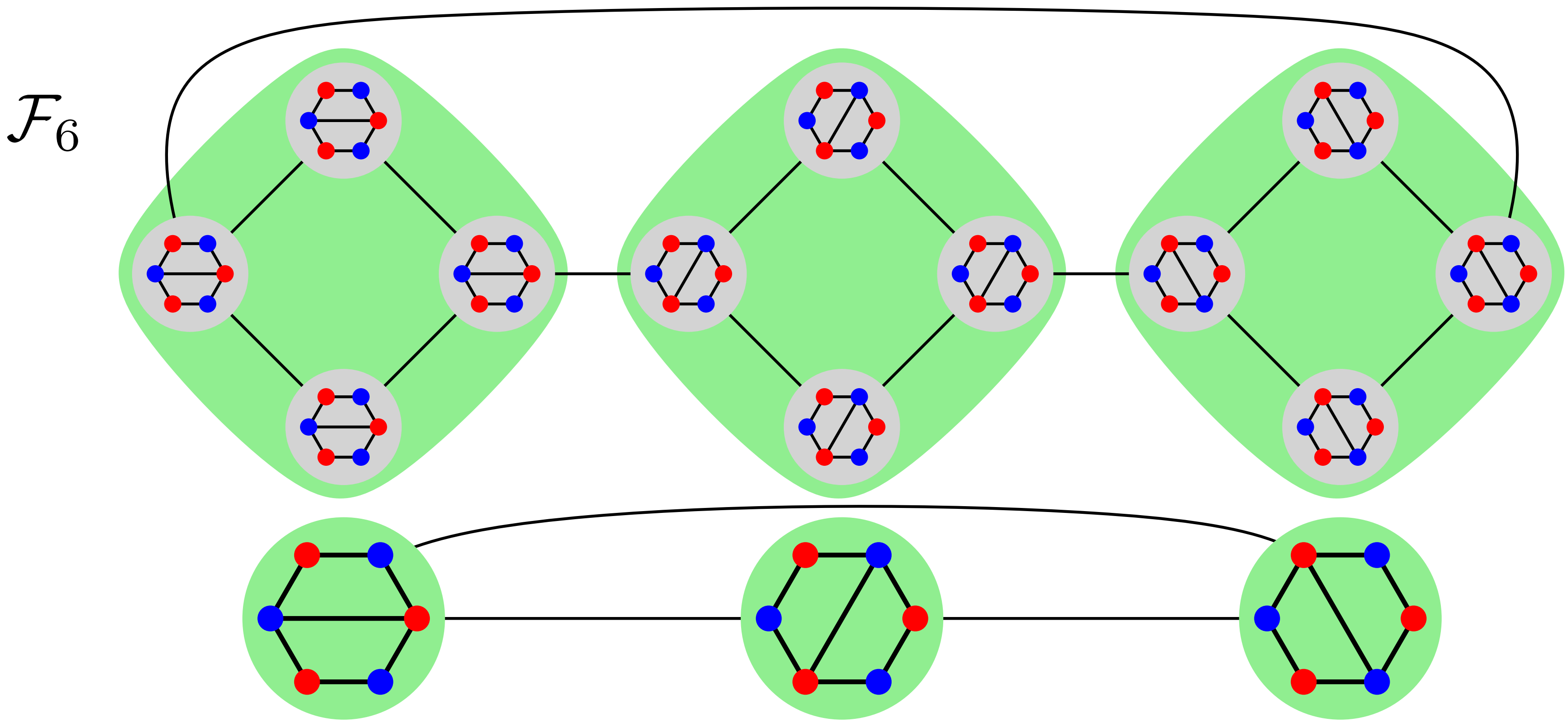

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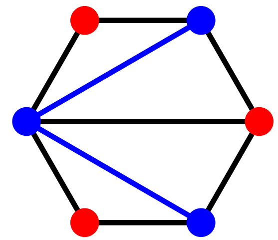

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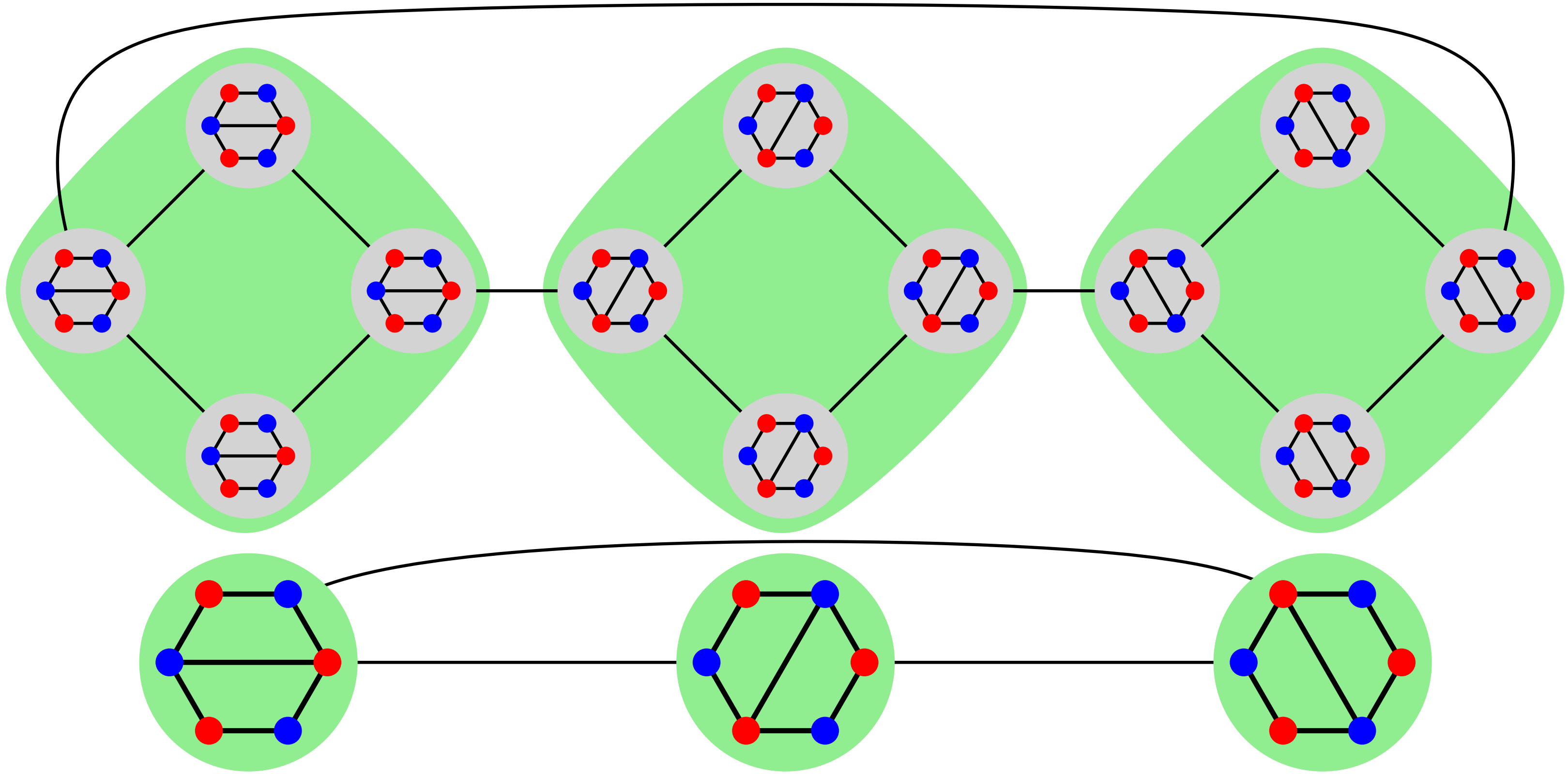

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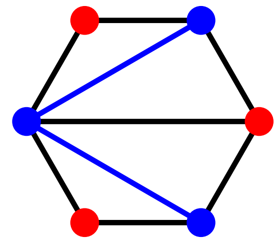
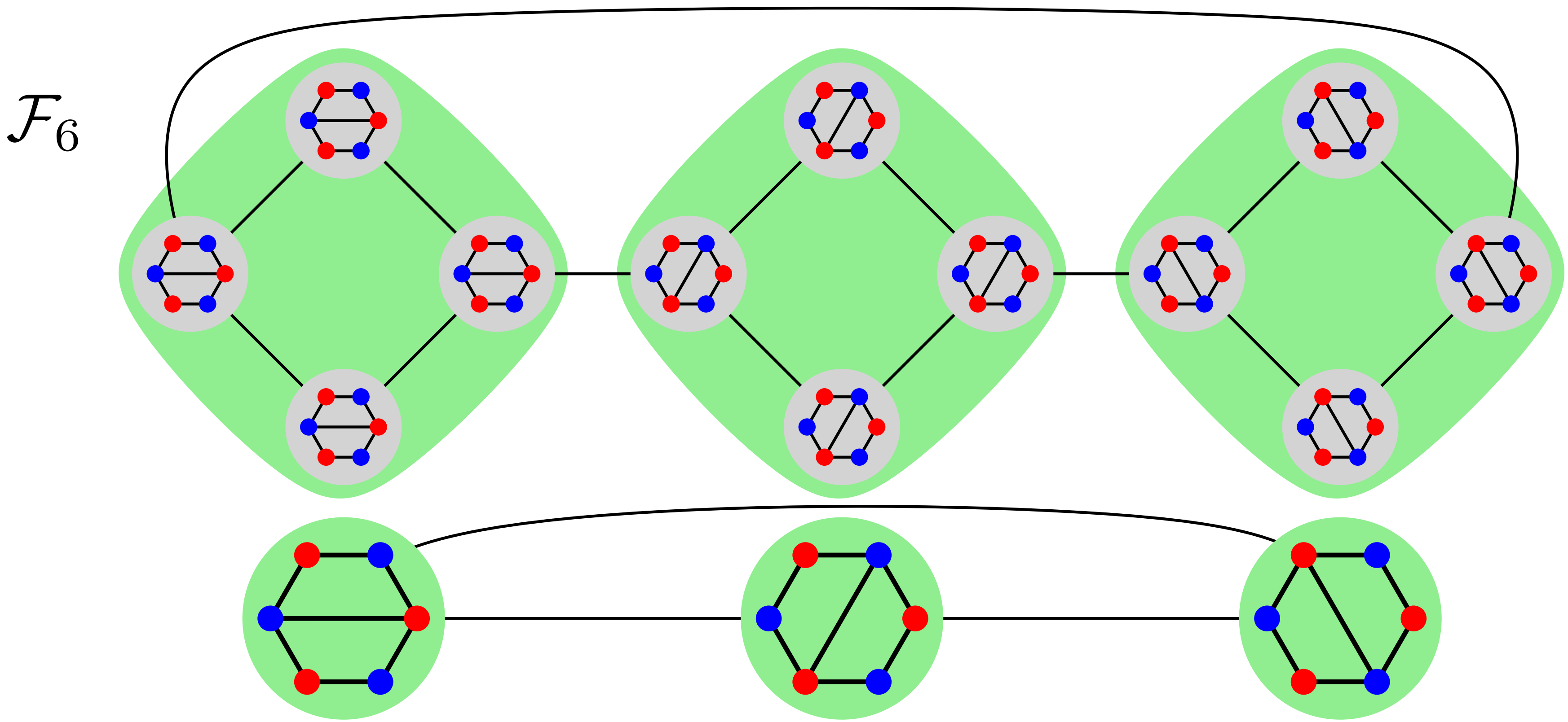
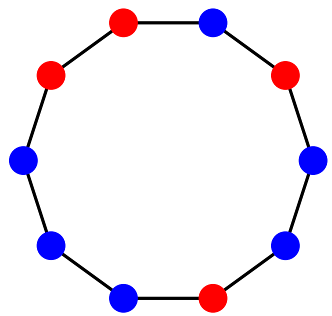




- Flip graph of colorful edges is rotation graph of **ternary** trees
- HC in $\mathcal{F}_N$: Combine Gray codes for ternary trees & hypercubes

A generalization

- Coloring sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, ℓ even



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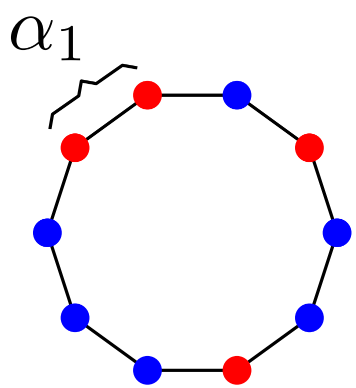

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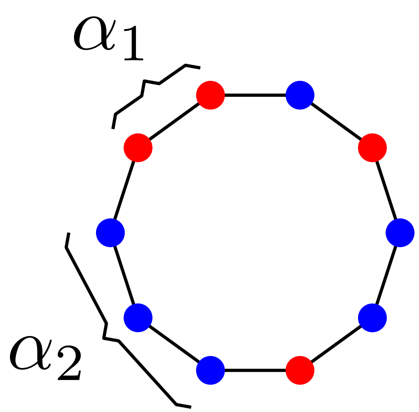

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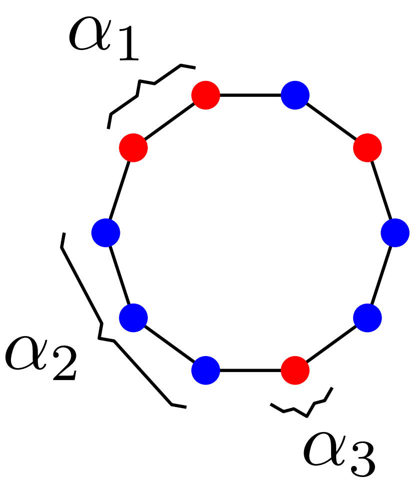

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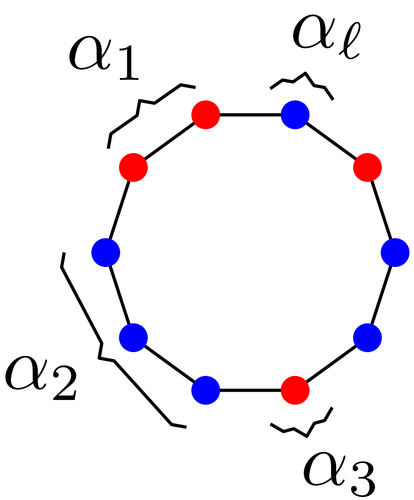

$$\alpha = (2, 3, 1, 2, 1, 1), \ell = 6$$

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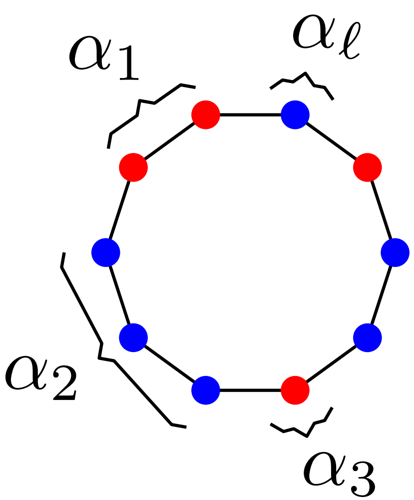

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ℓ = #blocks = #color changes

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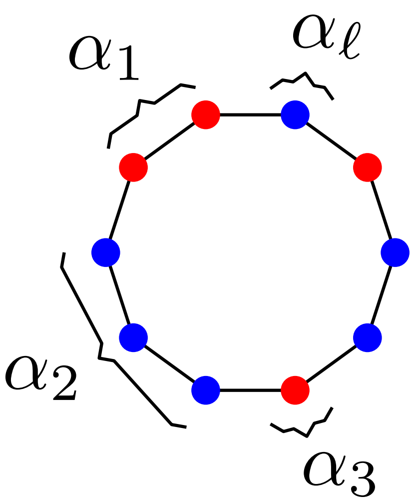

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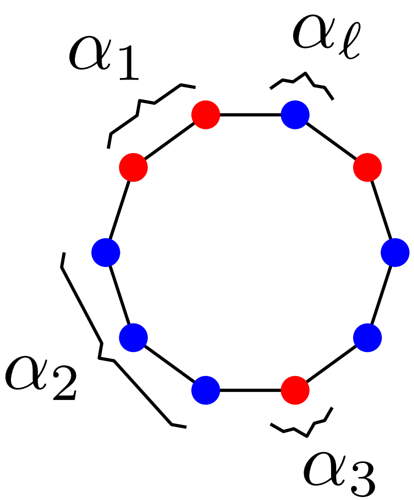

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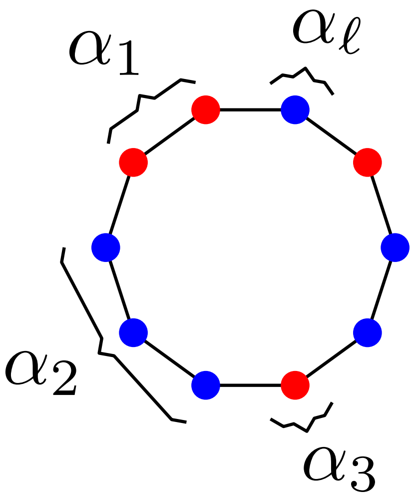

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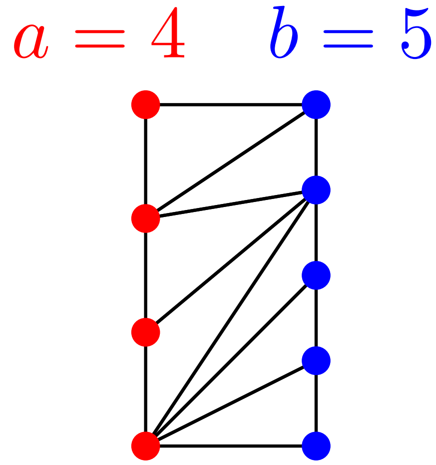
- **Theorem:** For every α of (even) length $\ell \geq 10$, the graph $\mathcal{F}_\alpha$ has a Hamilton cycle.

Two blocks



- $\alpha = (\alpha_1, \alpha_2) = (a, b)$

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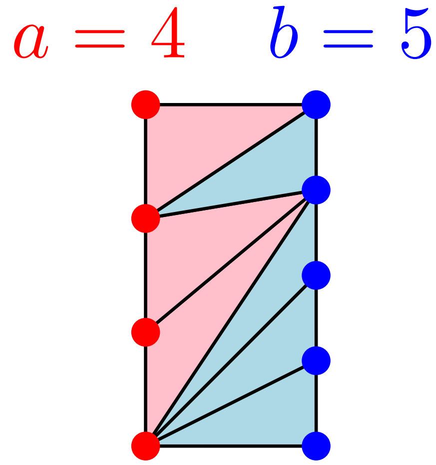

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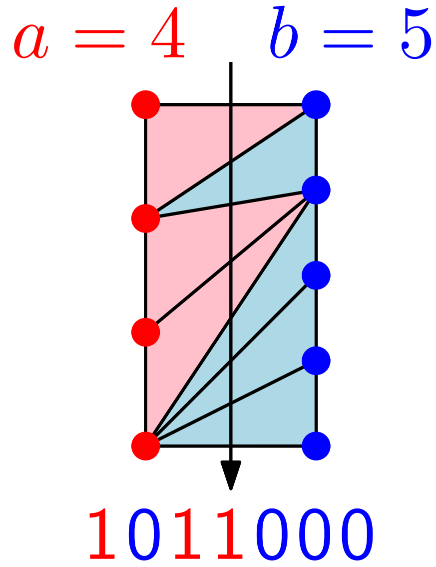

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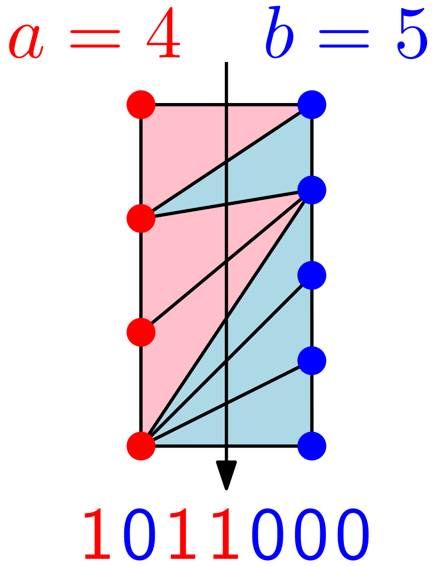

$(a-1, b-1)$-combination

Two blocks

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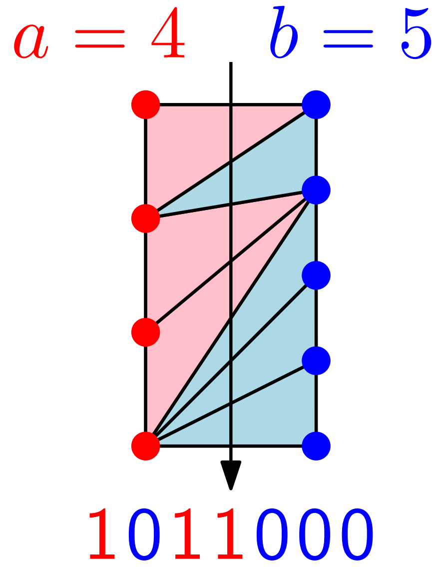

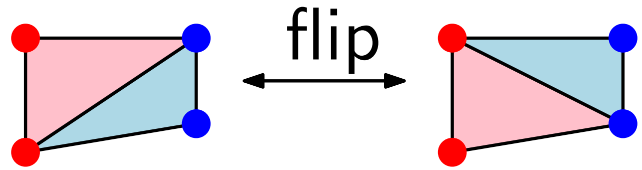

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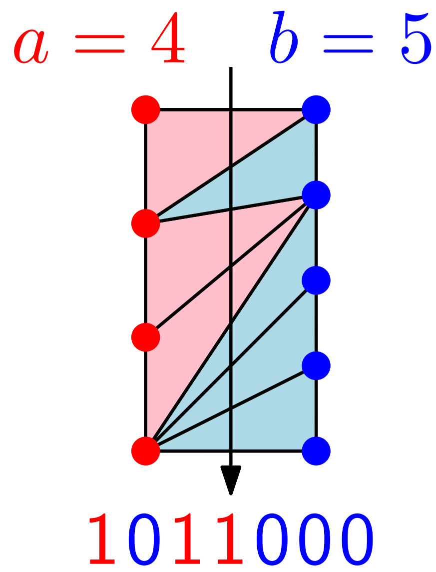

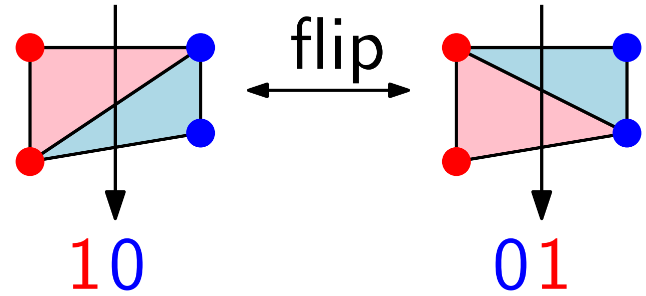

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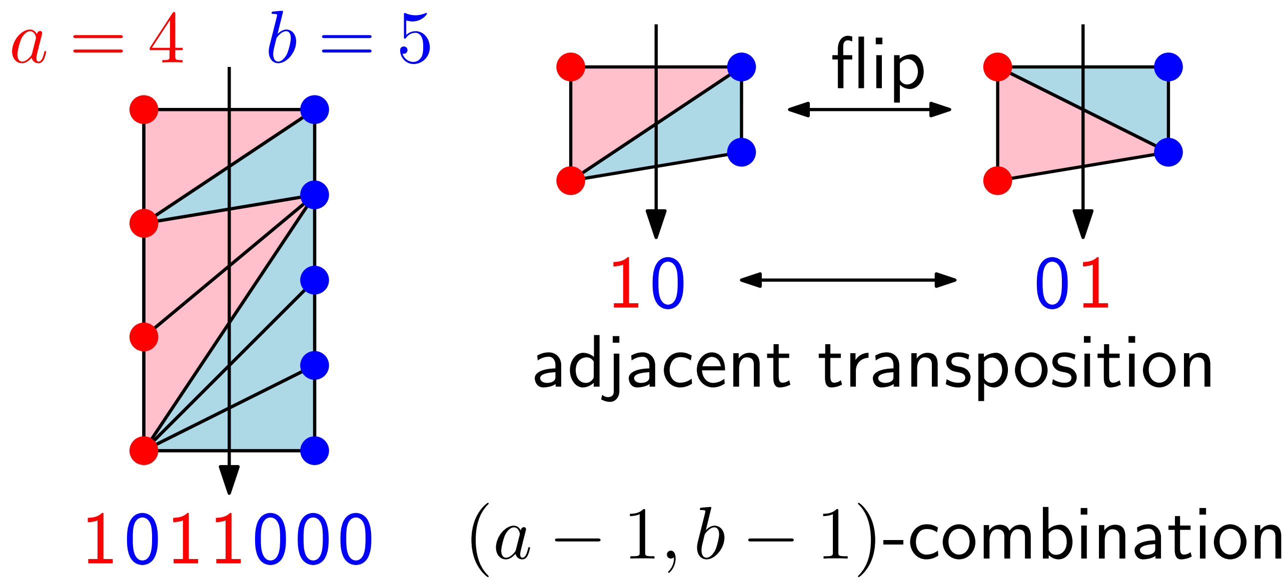

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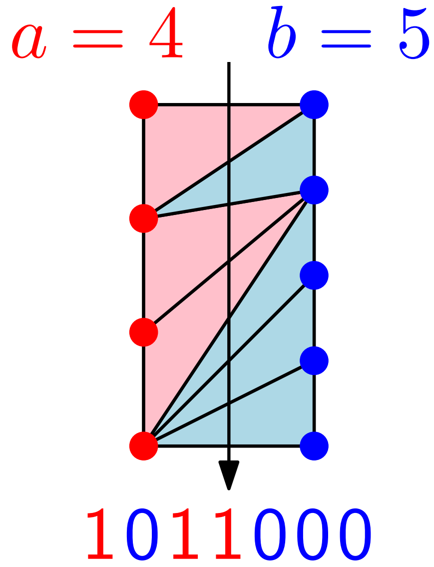

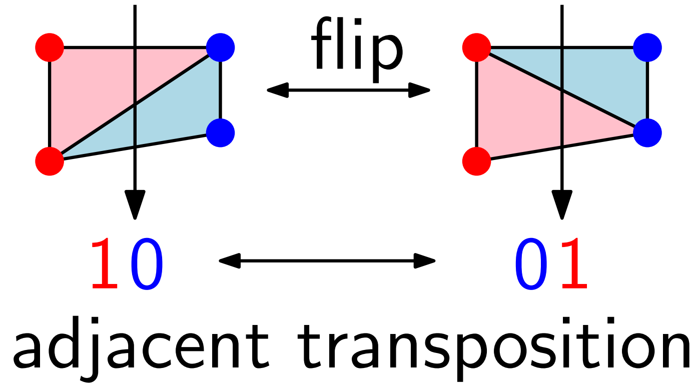

$(a-1, b-1)$-combination

- $\mathcal{F}_{(a,b)}$ is isomorphic to flip graph of $(a-1, b-1)$-combinations under adjacent transpositions

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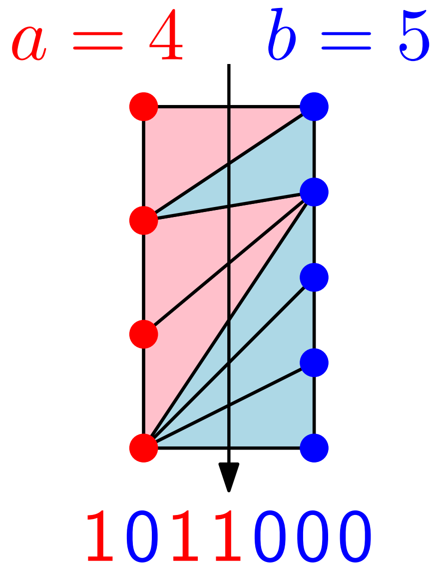

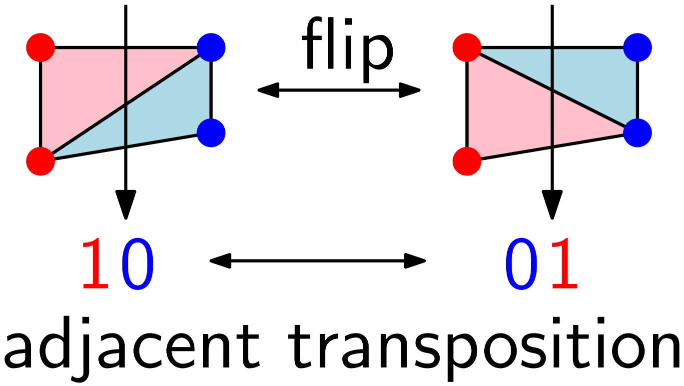

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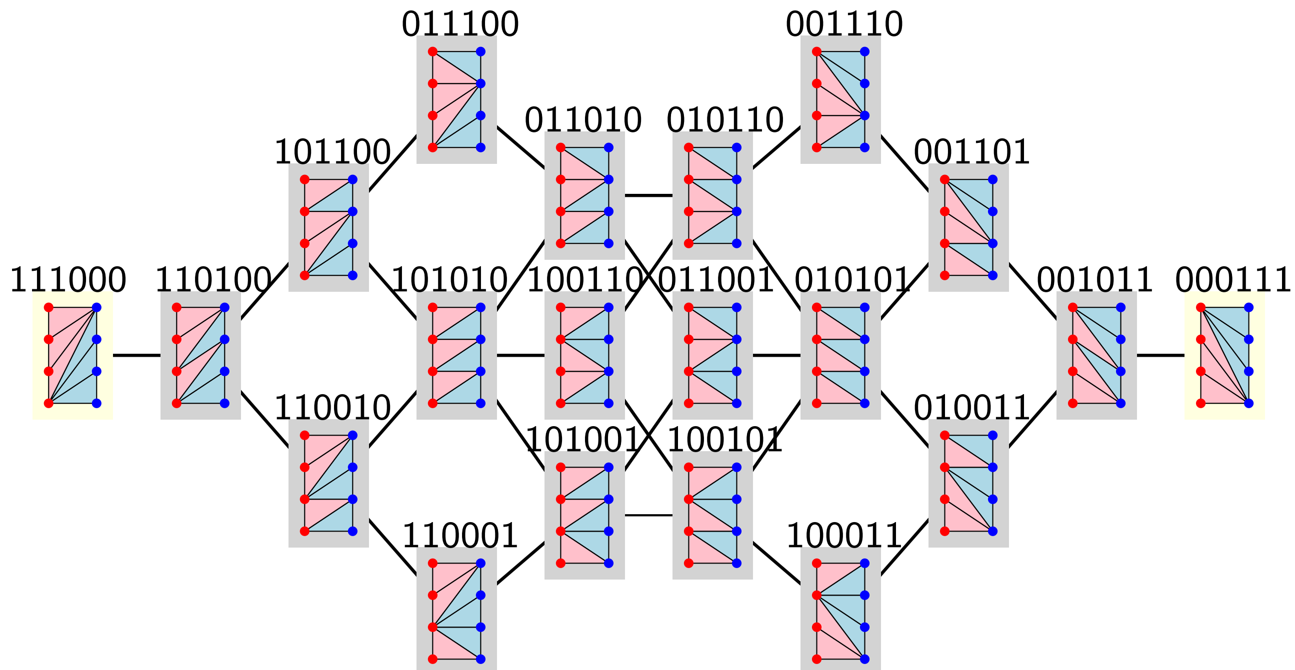
$\mathcal{F}_{(4,4)}$



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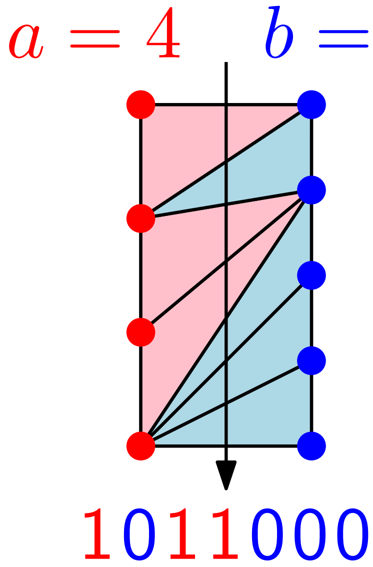

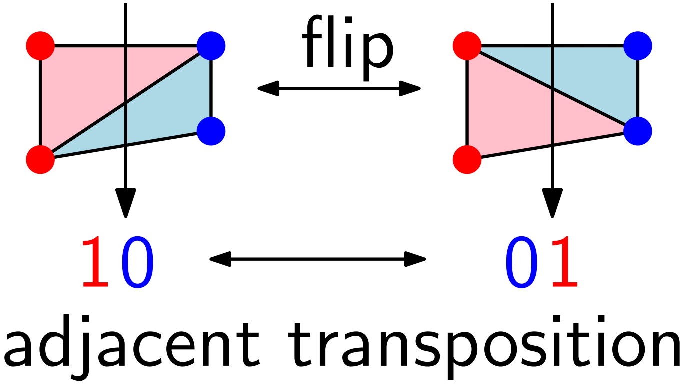

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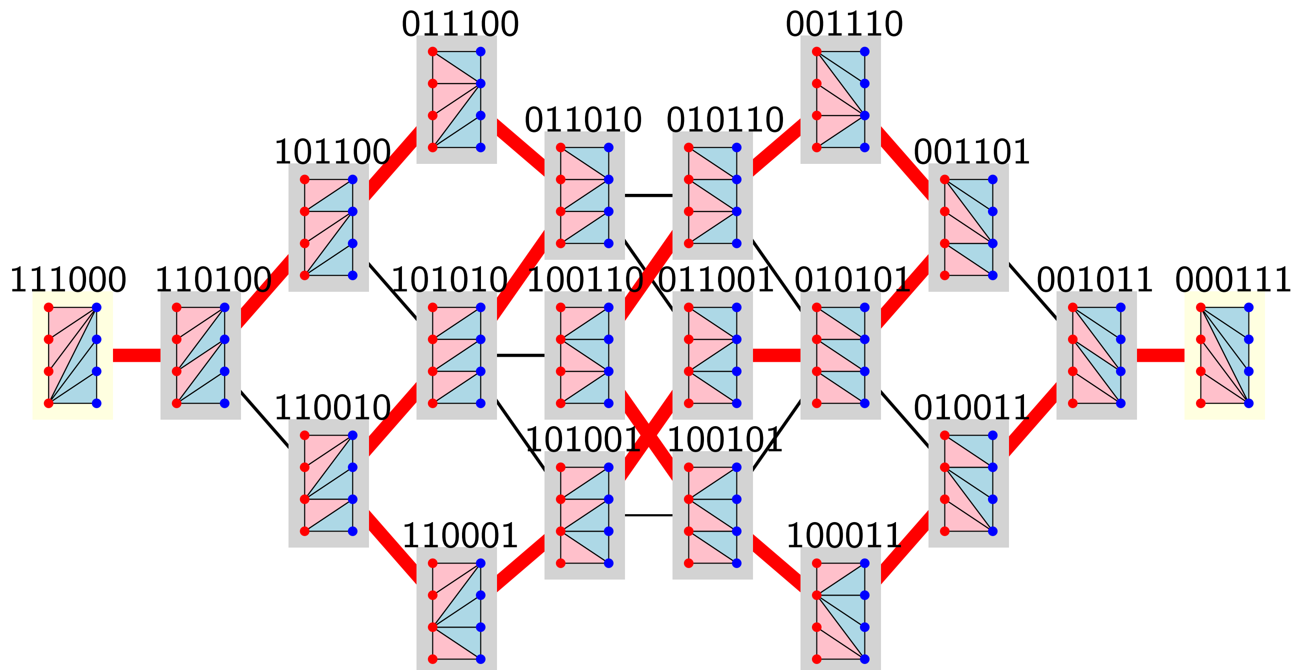

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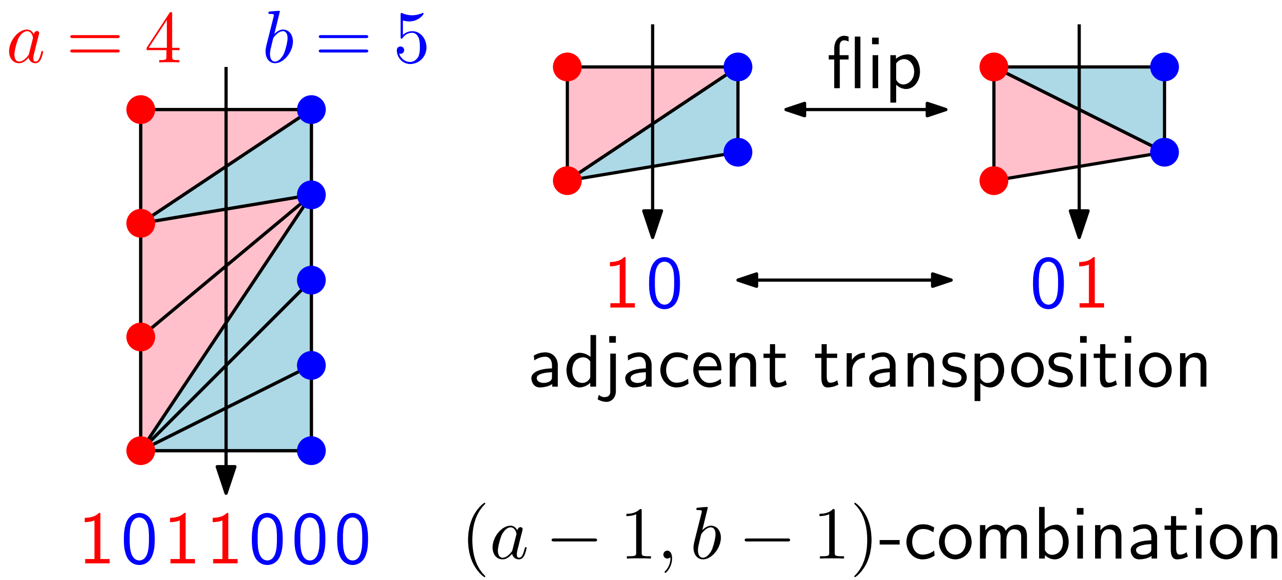

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- Hamiltonicity was solved by [Buck, Wiedemann 84], [Eades, Hickey, Read 84]

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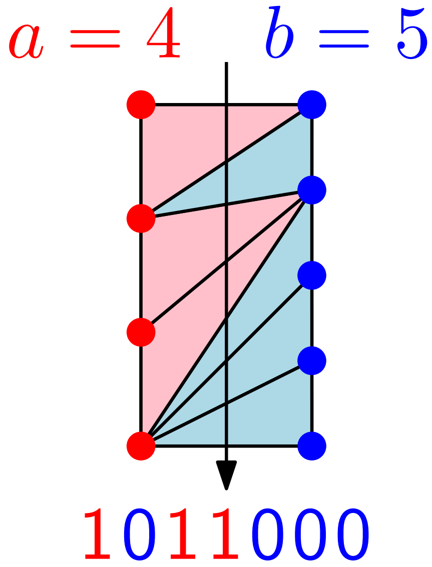
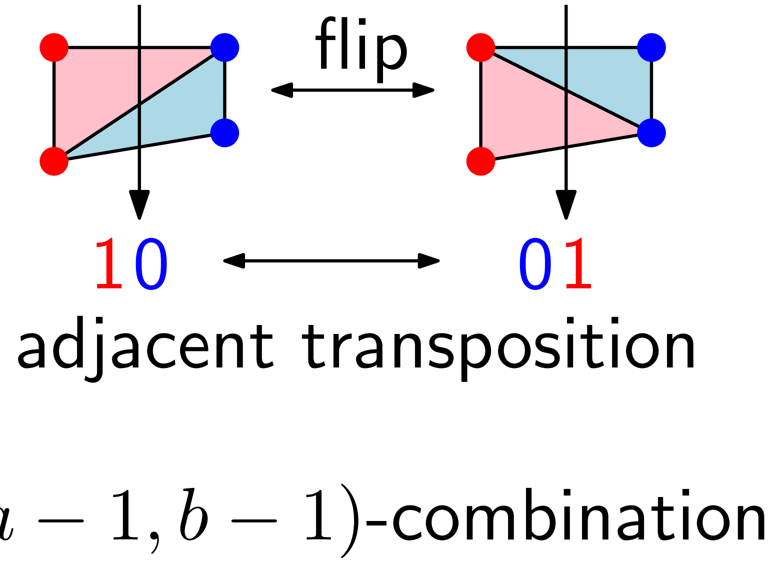
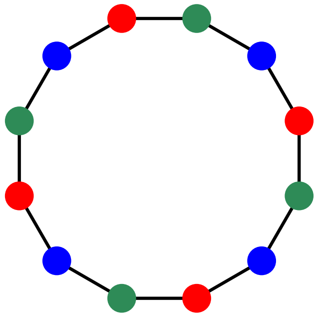




- $\mathcal{F}_{(a,b)}$ is isomorphic to flip graph of $(a-1, b-1)$-combinations under adjacent transpositions
- Hamiltonicity was solved by [Buck, Wiedemann 84], [Eades, Hickey, Read 84]
- **Theorem:** $\mathcal{F}_{(a,b)}$ has a Hamilton path iff $a \in \{1,2\}$ or $b \in \{1,2\}$ or a and b are both even.

Three colors

- Colors points red, blue, green alternatingly



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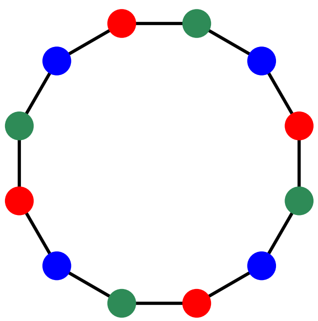
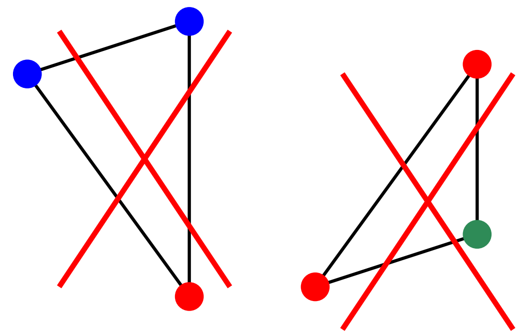


Want: every triangle
sees all three colors



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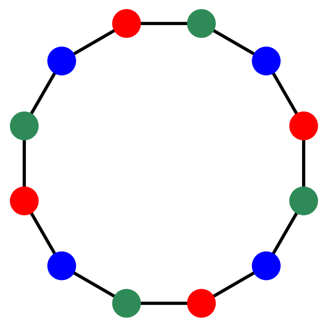

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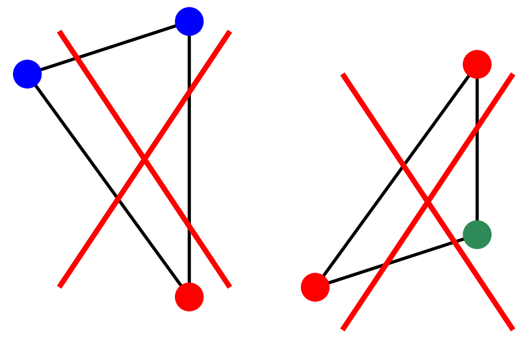

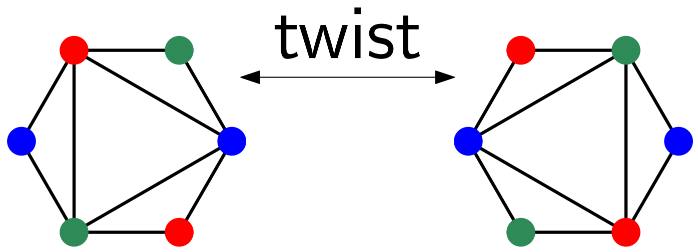

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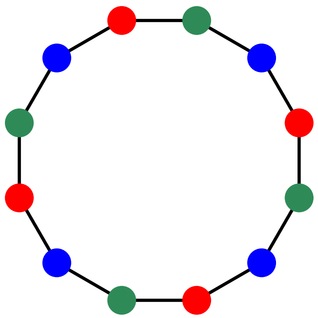

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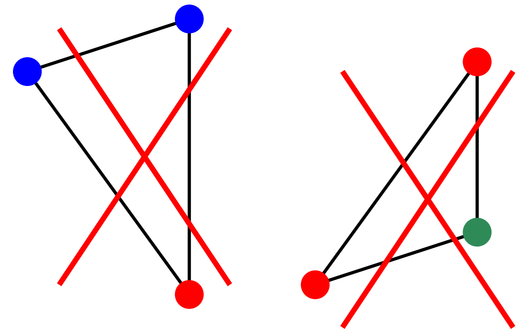

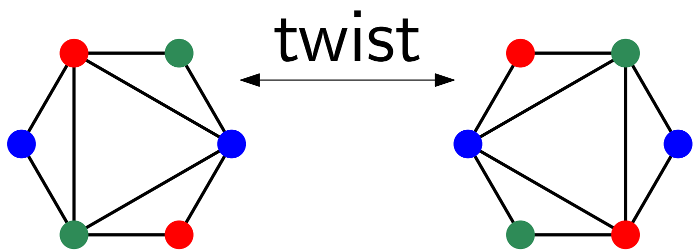

- Flip graph $\mathcal{H}_N$

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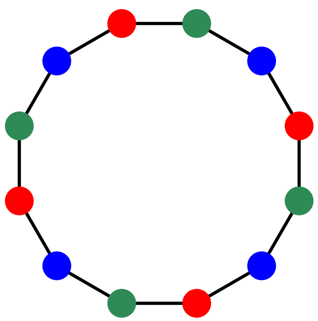

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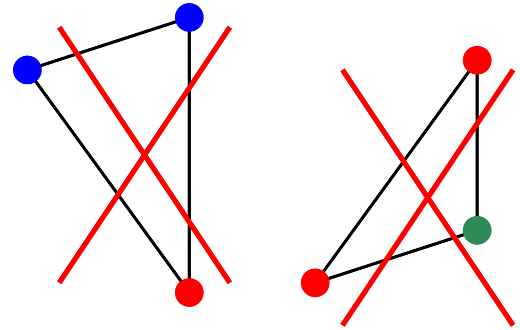
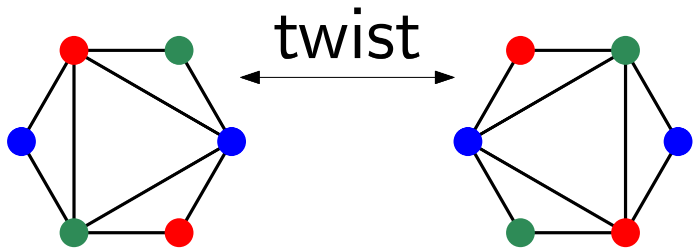
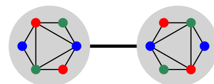
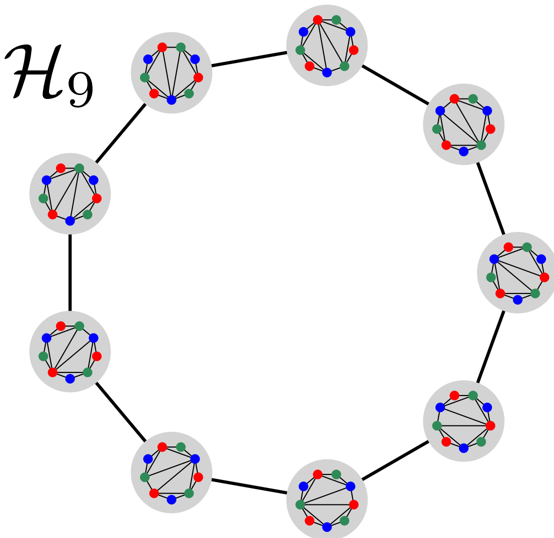




$\mathcal{H}_6$



$\mathcal{H}_9$



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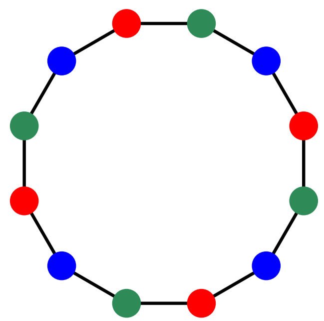

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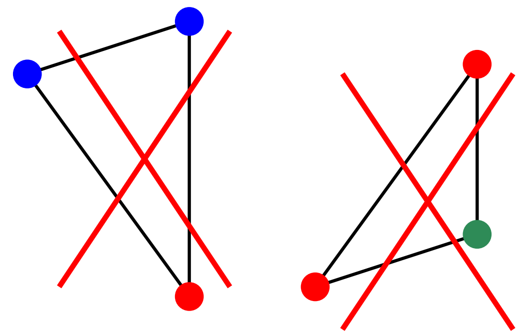

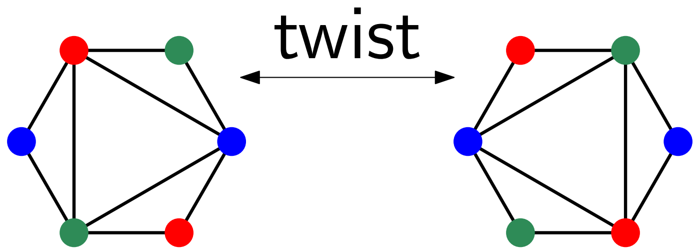

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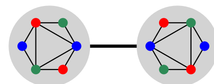

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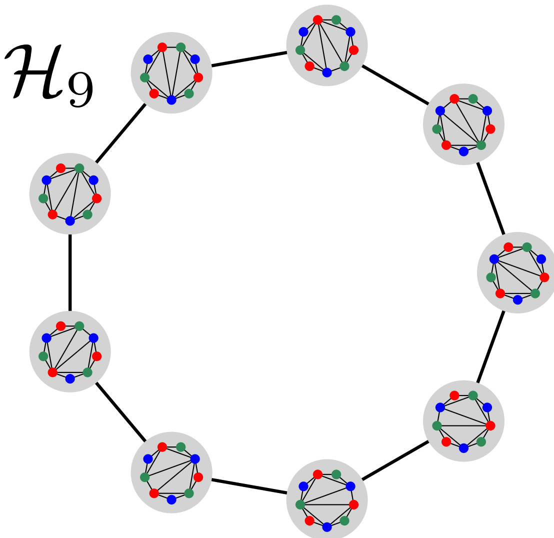

- Flip graph $\mathcal{H}_N$

- **Theorem:** For any N a multiple of 3, $\mathcal{H}_N$ is connected.

Open problems

- Hamiltonicity of $\mathcal{H}_N$ when $3|N$ (3 colors + twists)?

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- Possible cycle lengths in associahedron $\mathcal{G}_N$ apart from HCs?

Thank you!