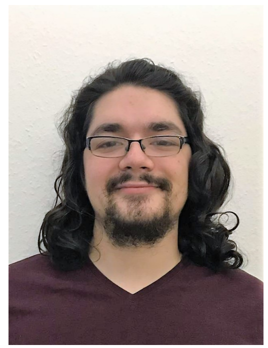
Kneser graphs are Hamiltonian

Torsten Mütze (Warwick + Prague)

joint with Arturo Merino (TU Berlin) and Namrata (Warwick)

extended abstract in [STOC 2023]



Introduction

- **Kneser graph** $K(n,k)$

  vertices $= \binom{[n]}{k}$

  edges $=$ pairs of disjoint sets

  $$A \cap B = \emptyset$$

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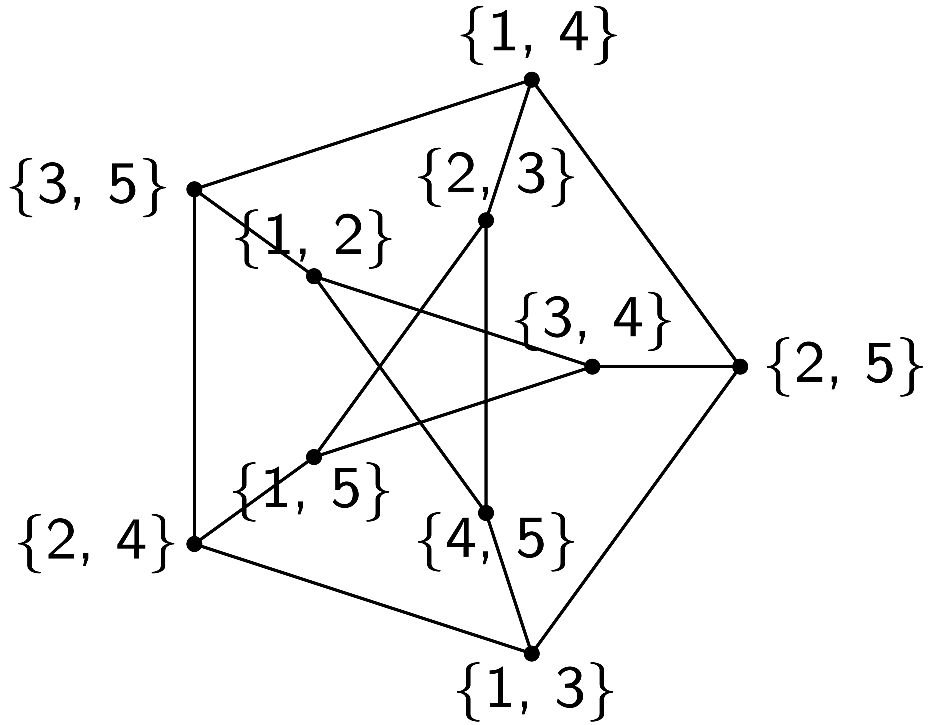

Petersen graph $K(5,2)$

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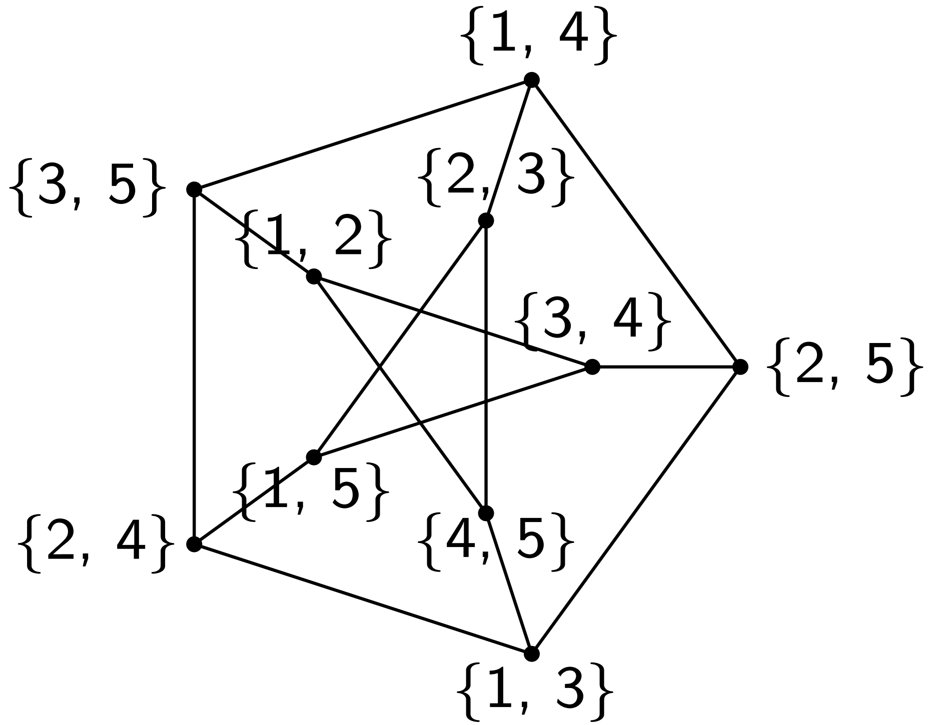

Petersen graph $K(5,2)$

- we assume $k \geq 1$ and $n \geq 2k+1$ (otherwise trivial)

Basic properties

- [Lovász 1978]: proof of Kneser's conjecture

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- [Erdős, Ko, Rado 1961]:

  $$\alpha(K(n,k)) = \binom{n-1}{k-1}$$

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- Kneser graphs: should be easier for dense cases

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  uses Baranyai, Kruskal-Katona, Ray-Chaudhuri-Wilson

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JOURNAL OF COMBINATORIAL THEORY (B) 20, 62–63 (1976)

The Rugby Footballers of Croam

MICHAEL MATHER

Department of Mathematics, University of Otago, Dunedin, New Zealand

Communicated by W. T. Tutte

Received November 7, 1974

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REFERENCES

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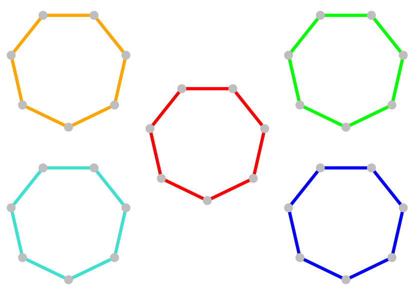

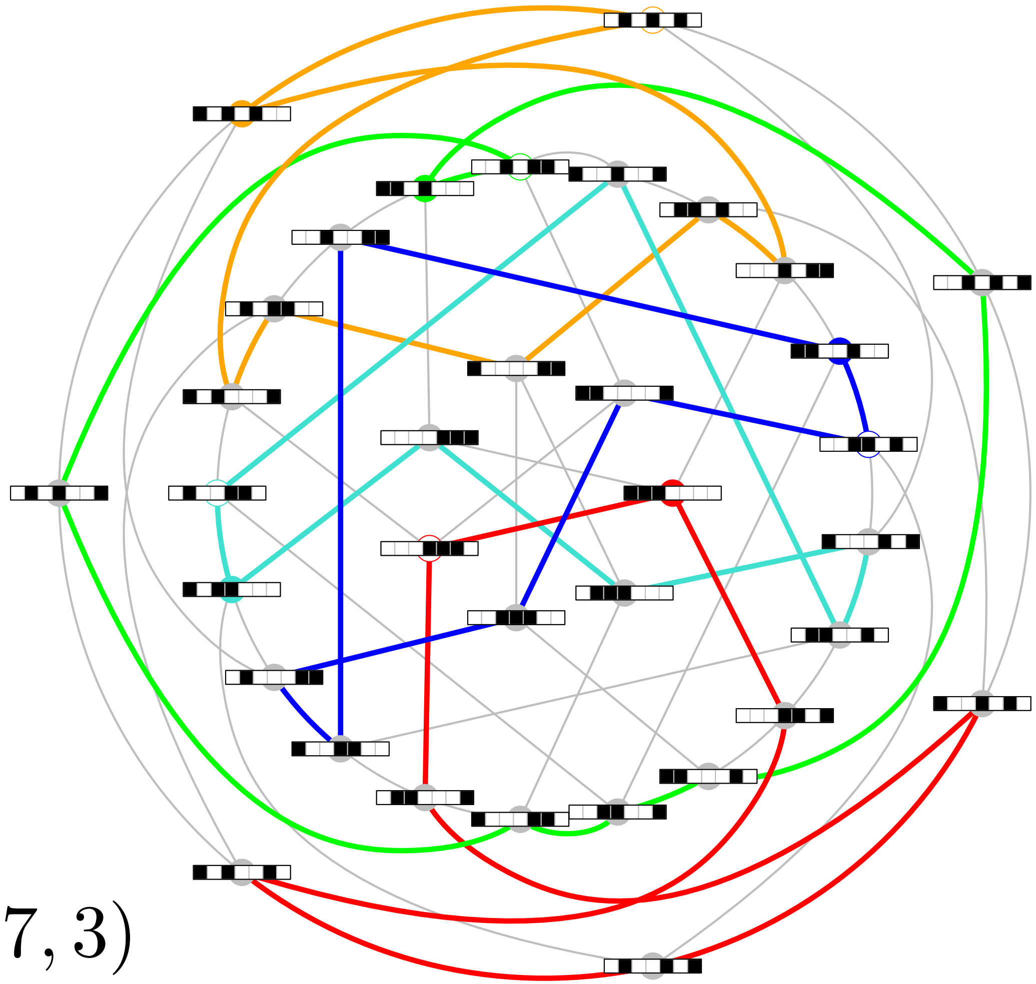

$O_3 = K(7,3)$

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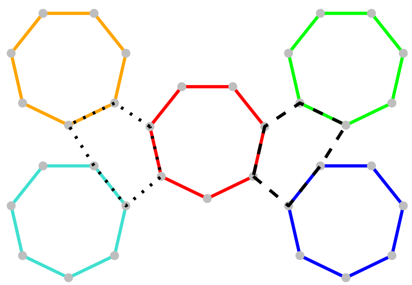

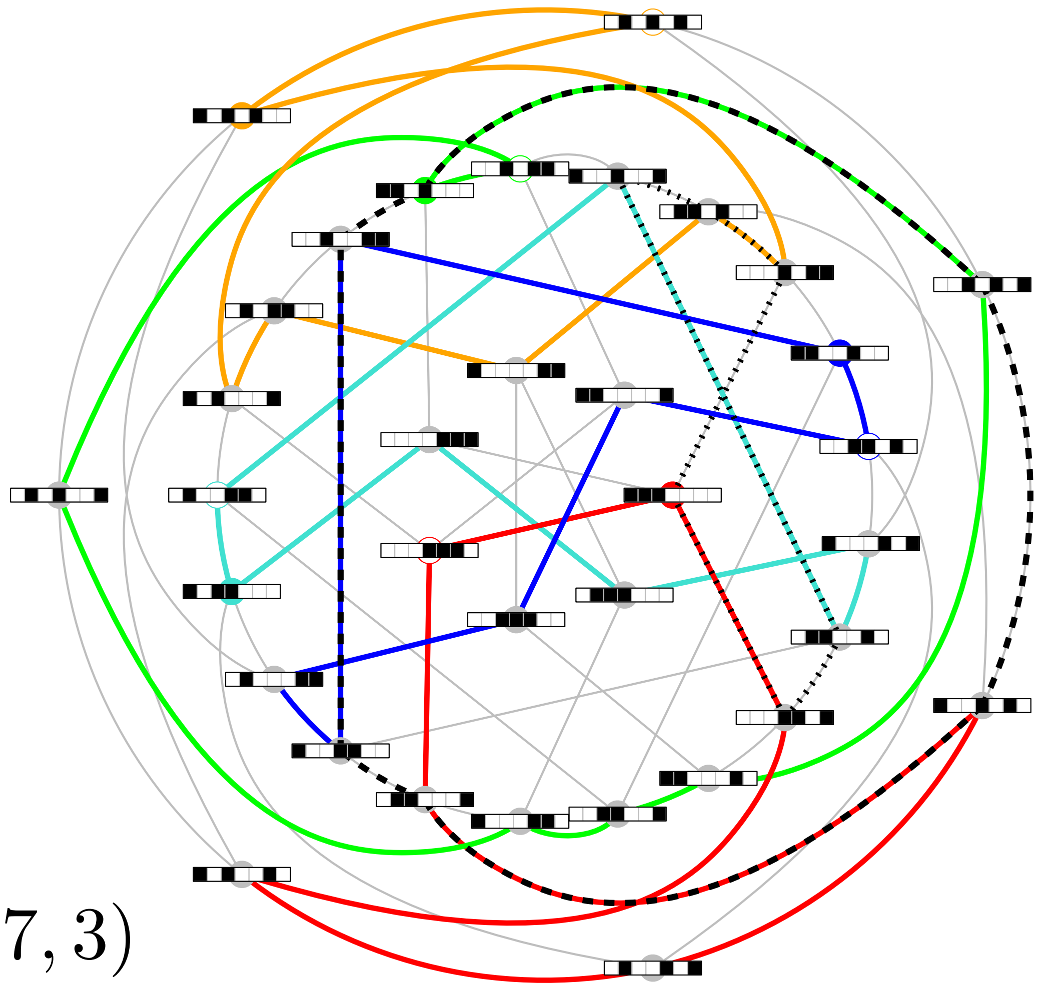

$O_3 = K(7,3)$

Hamilton cycles: sparse cases

- **Theorem** [M., Nummenpalo, Walczak 2021 JLMS]:
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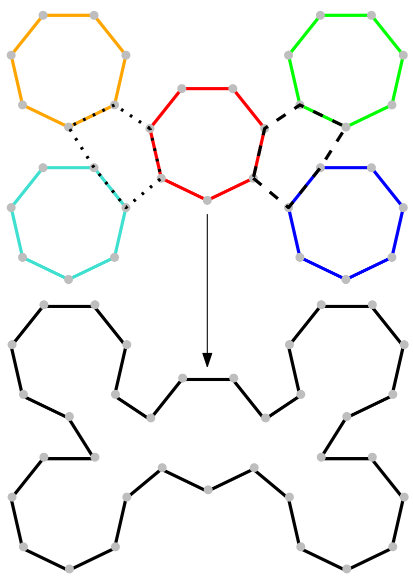

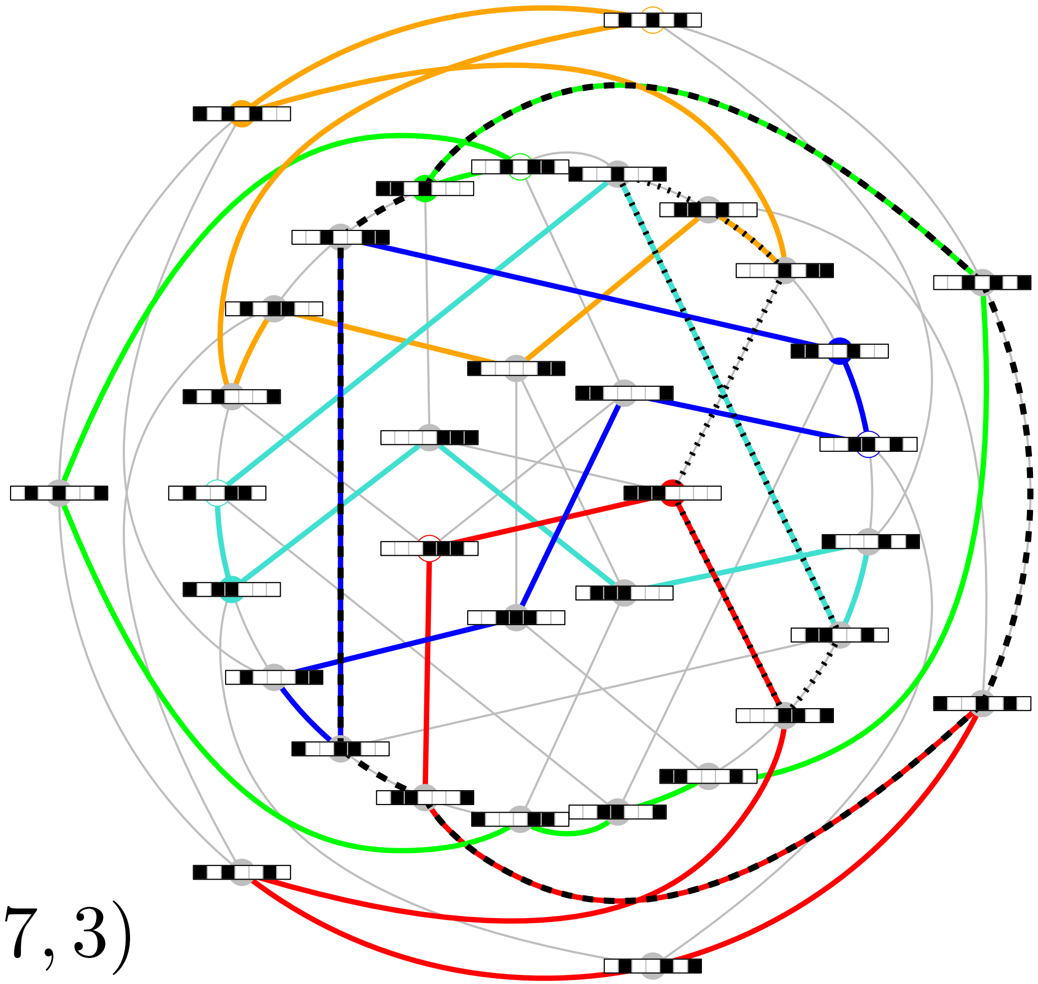

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Our results

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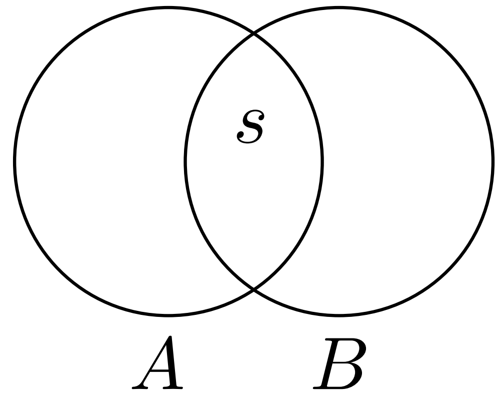

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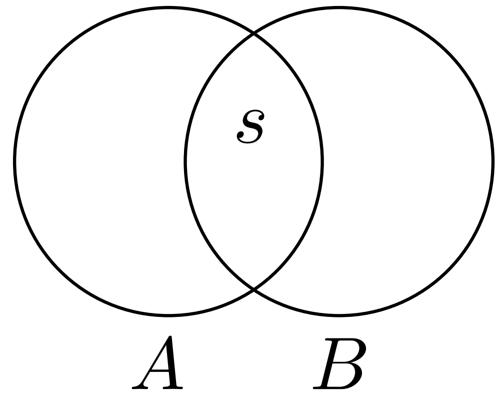

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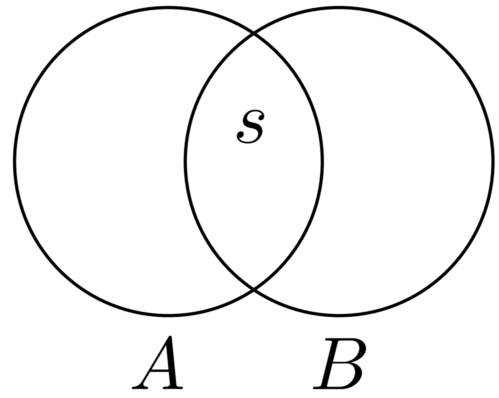

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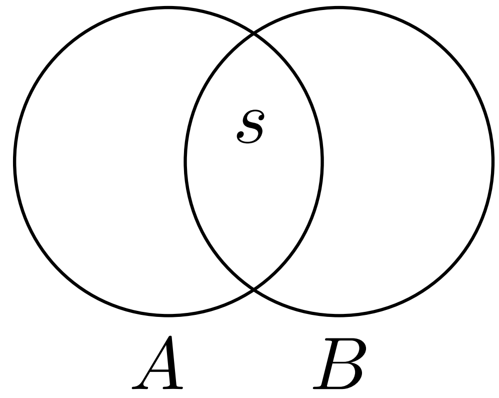

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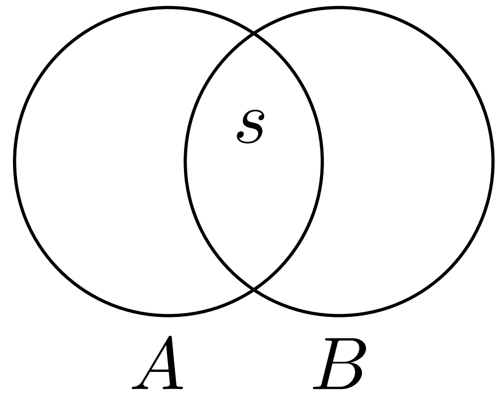

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- [Jiang, Ruskey 1994], [Knor 1994] proved that
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Our results

- **Theorem 2**:
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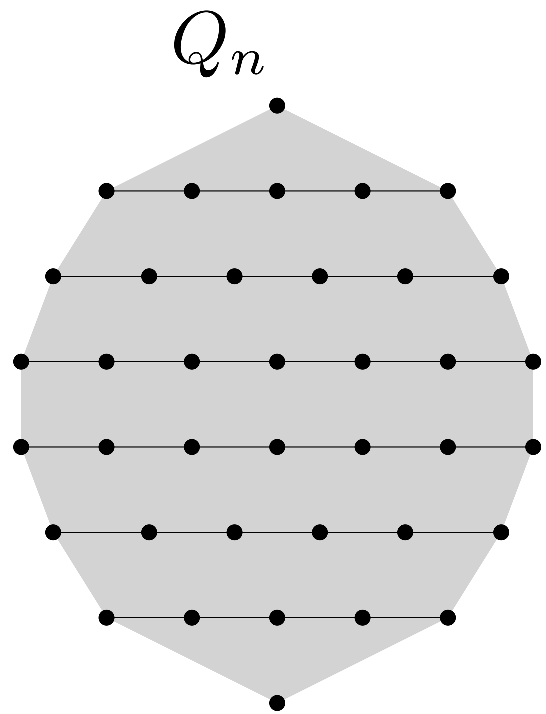

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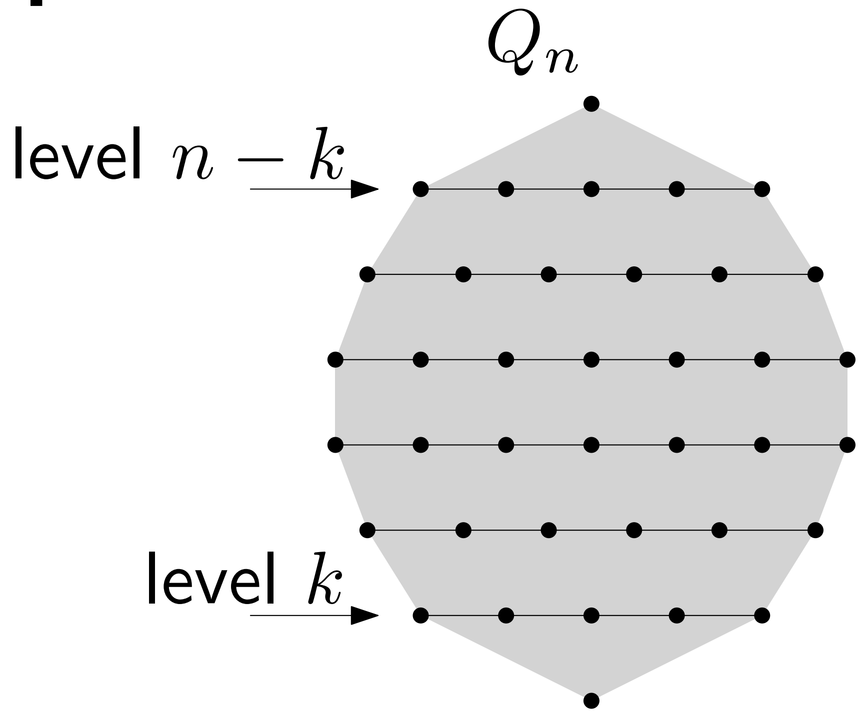

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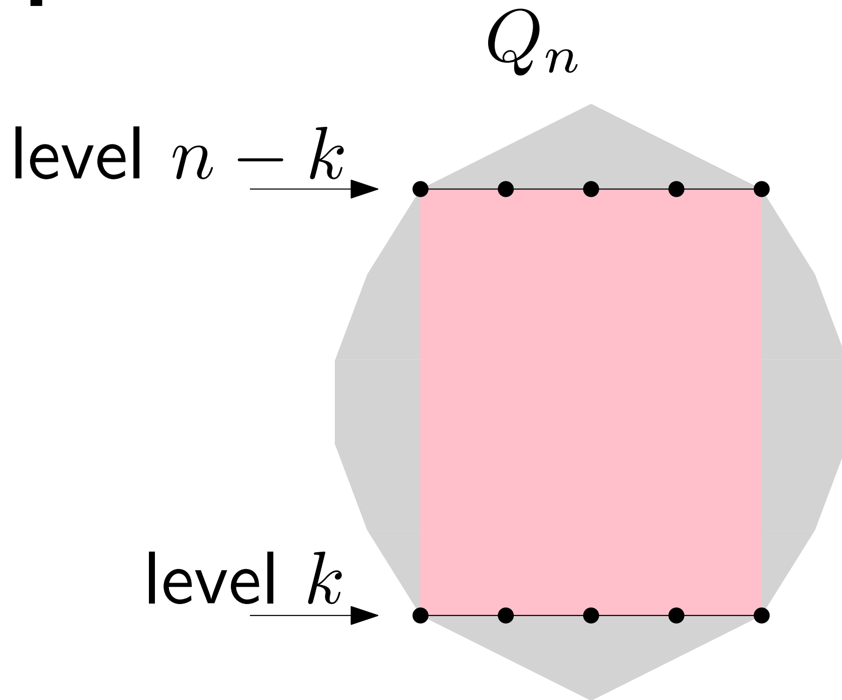

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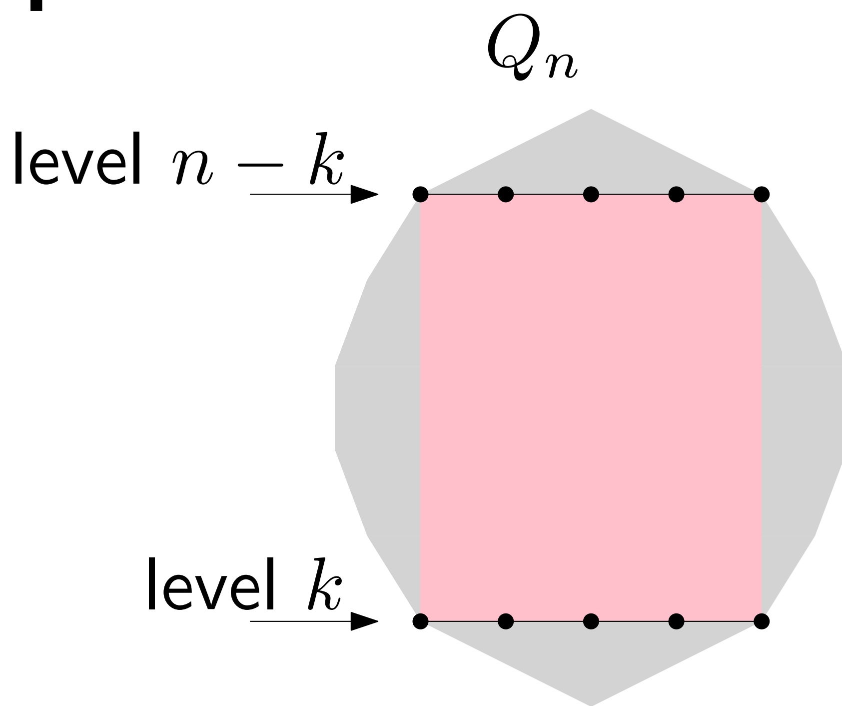

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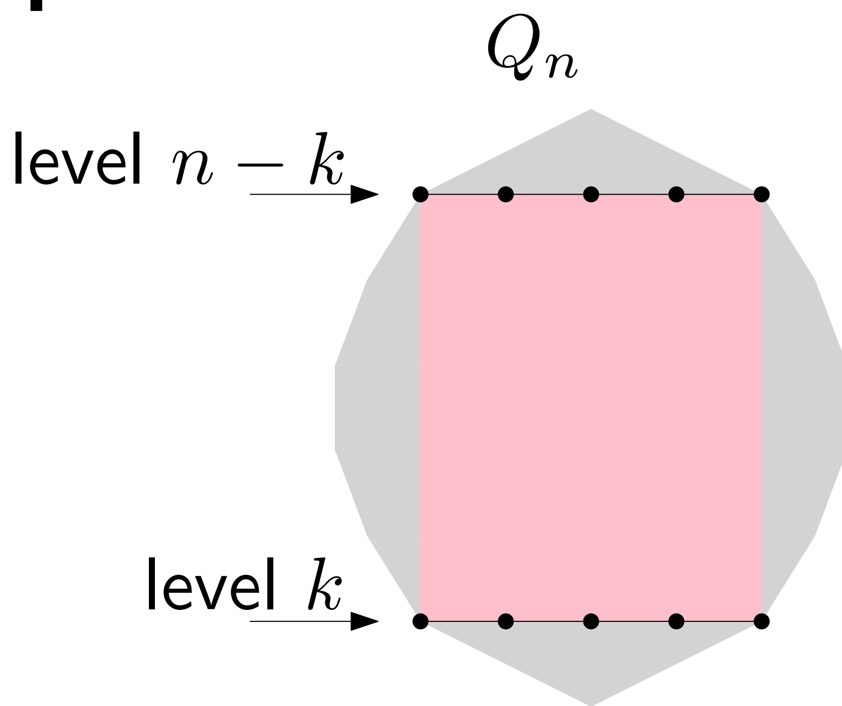

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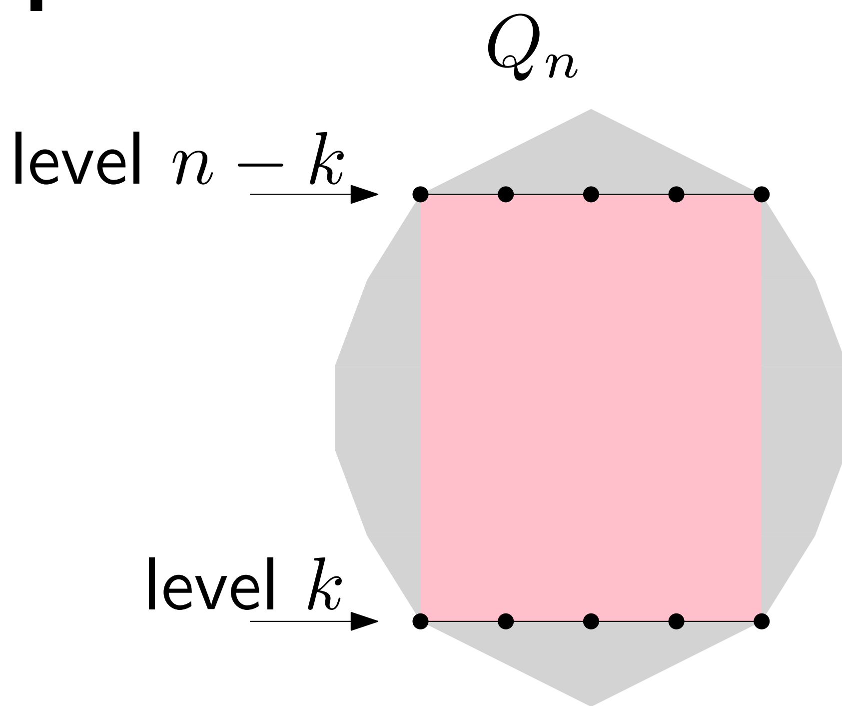

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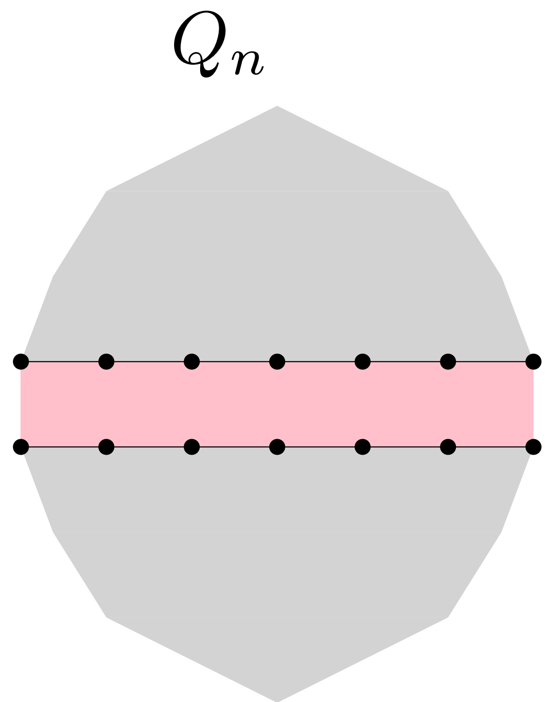

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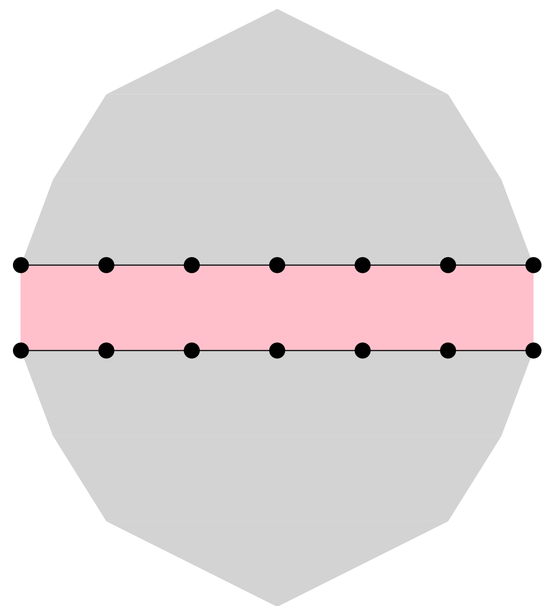

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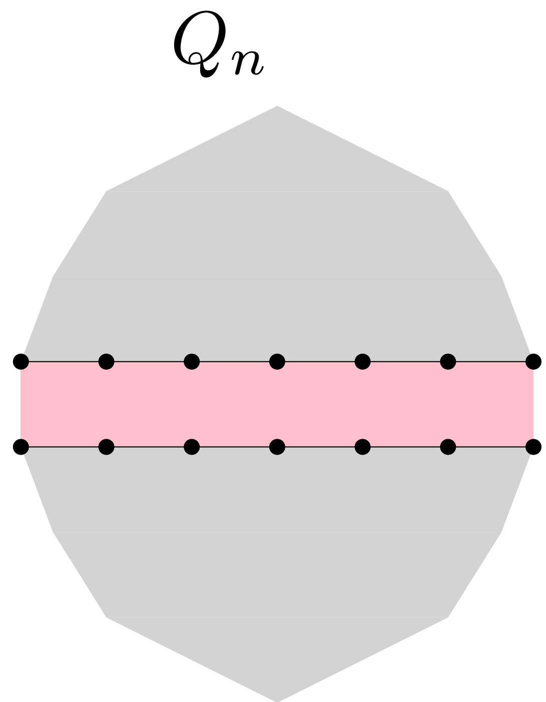

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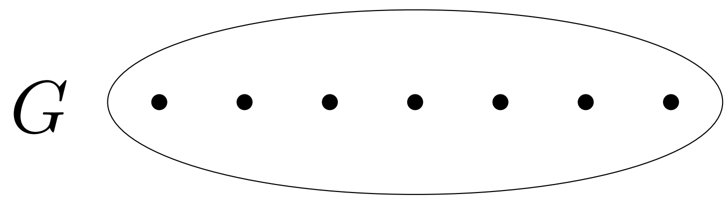

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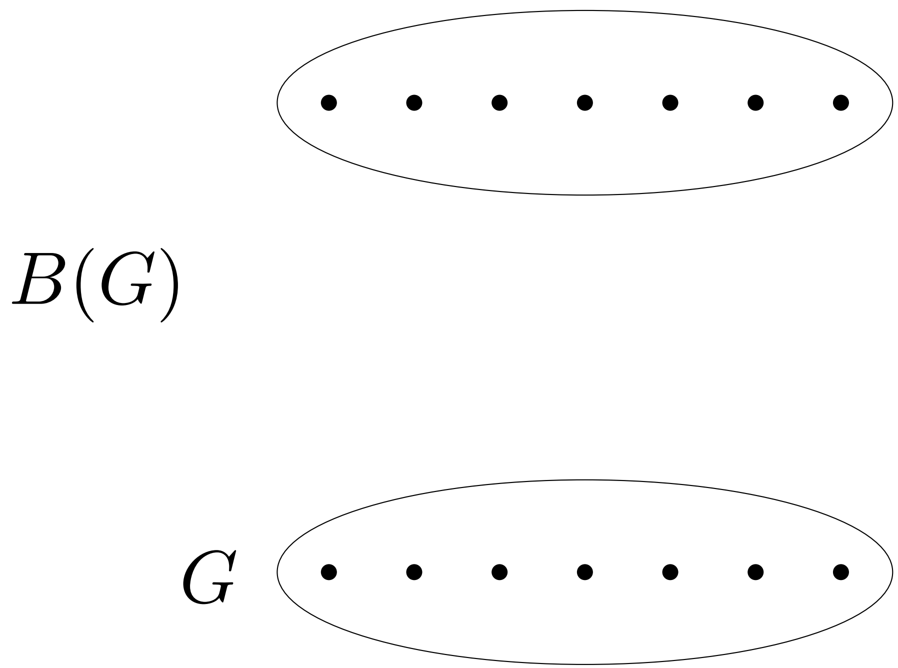

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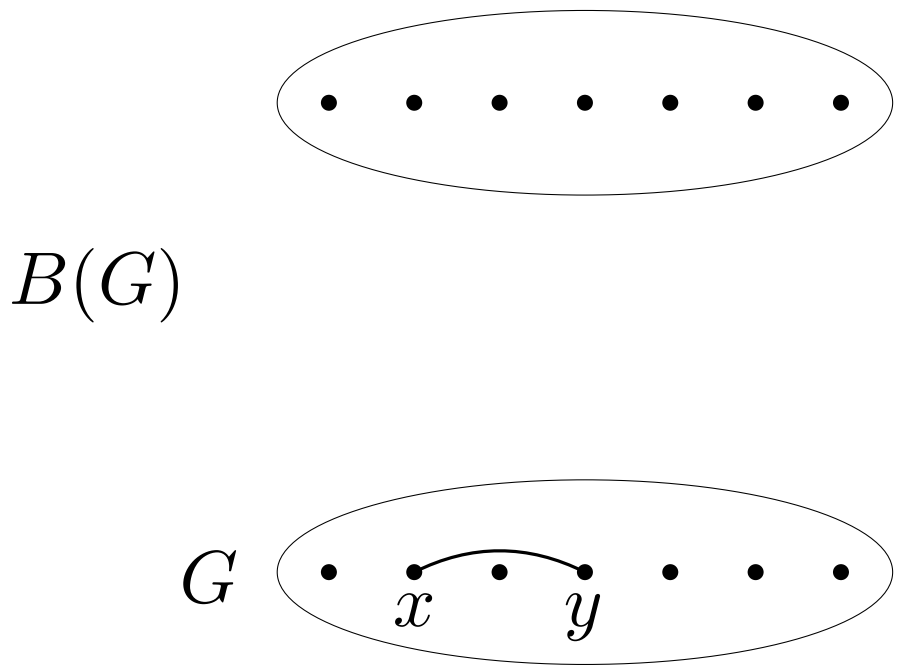

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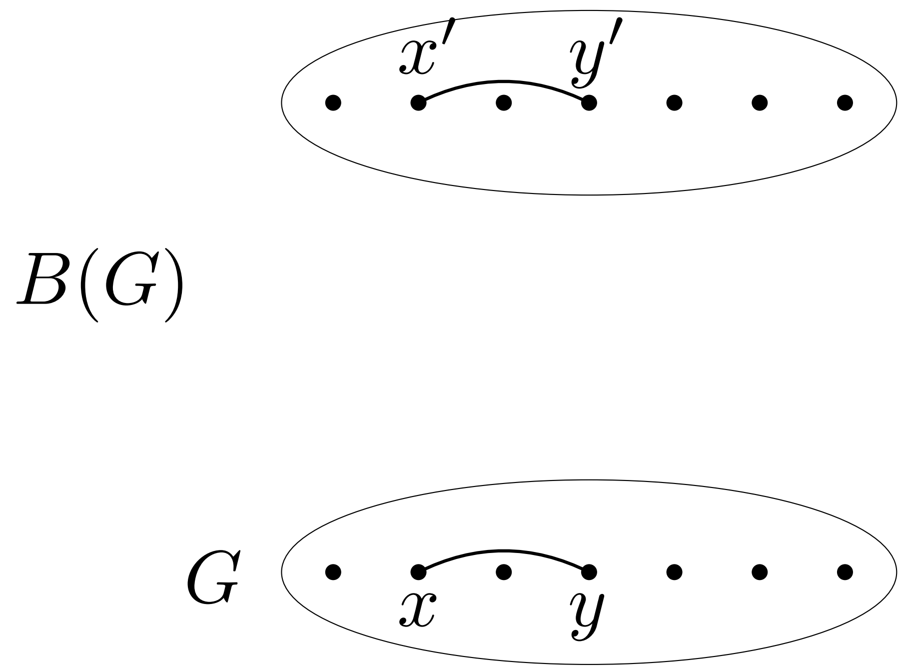

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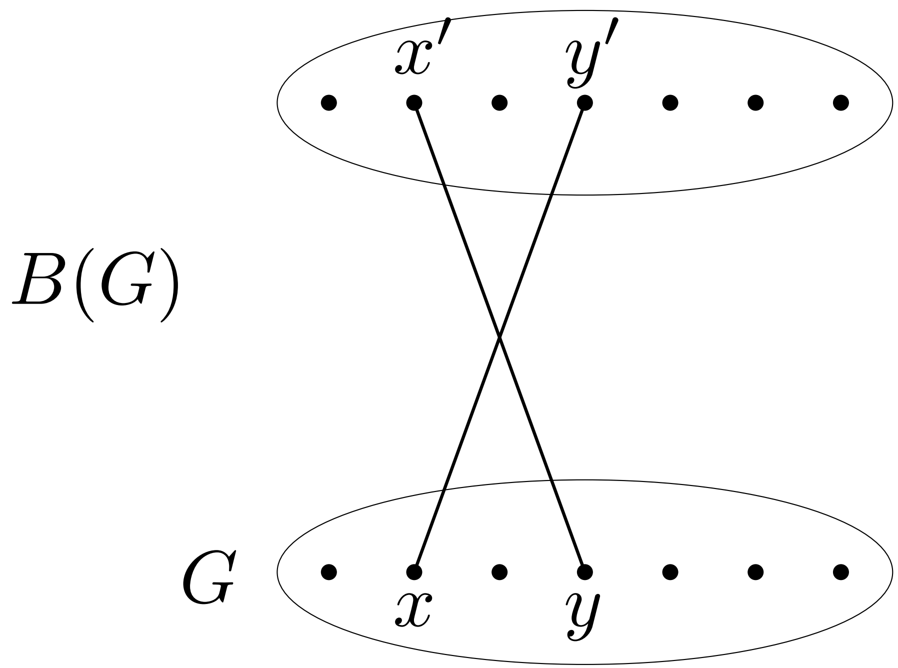

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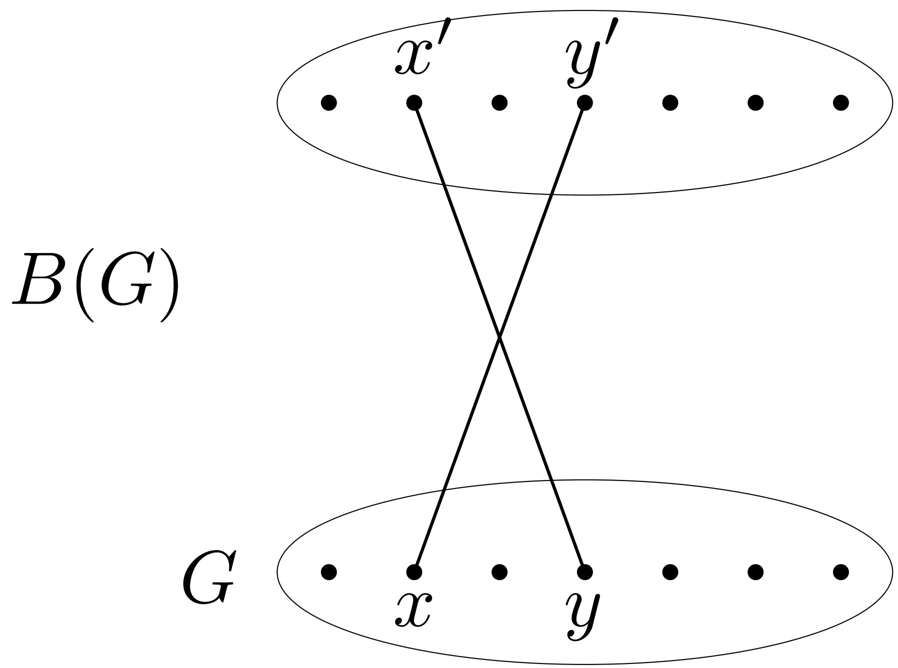

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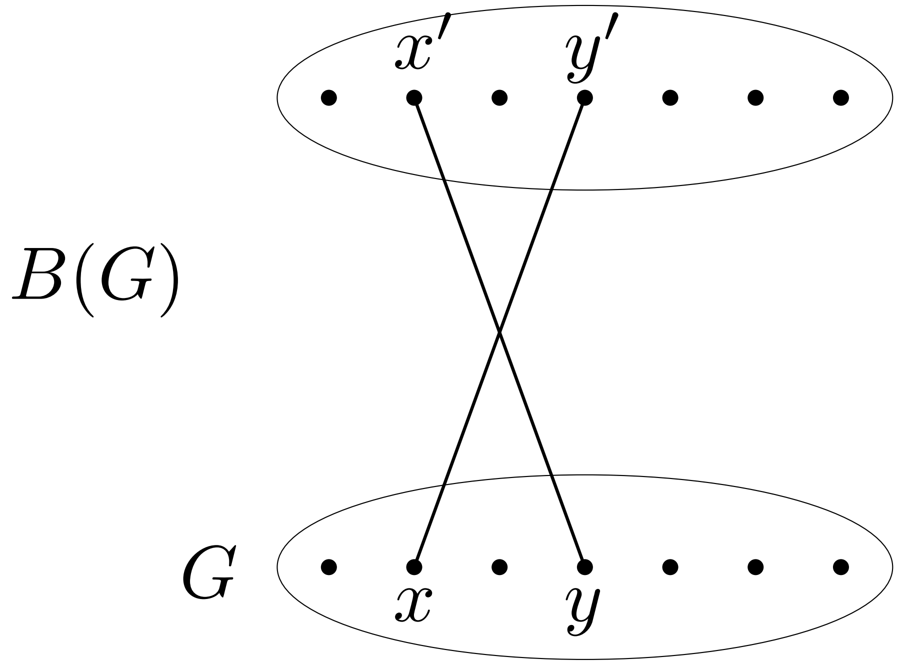

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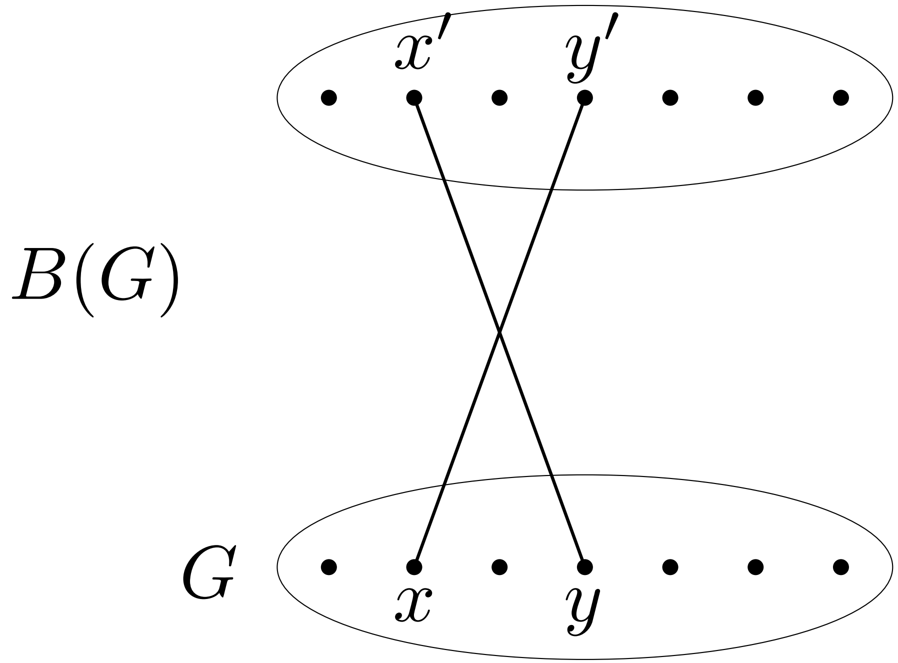
- we thus obtain a new proof for Hamiltonicity of $H(n, k)$



Summary of old and new results

Kneser graphs
$K(n,k)$

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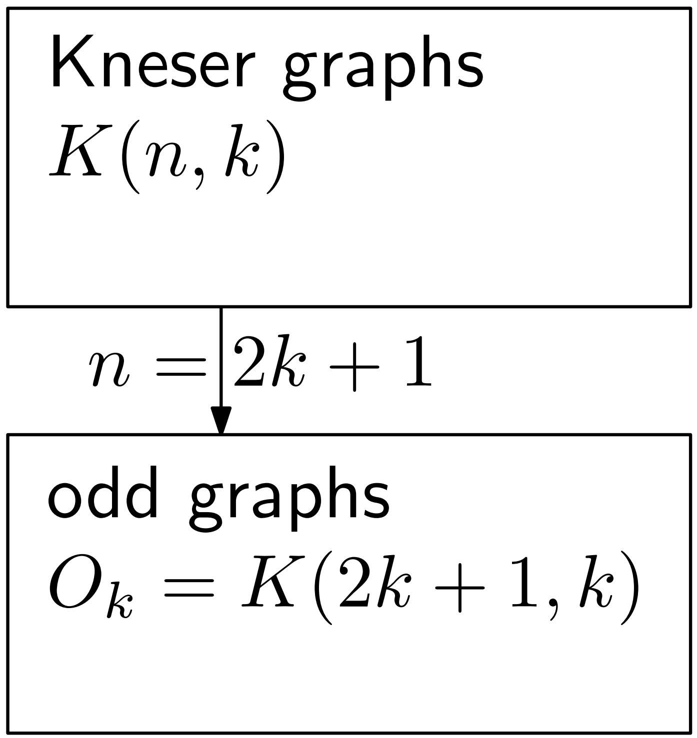

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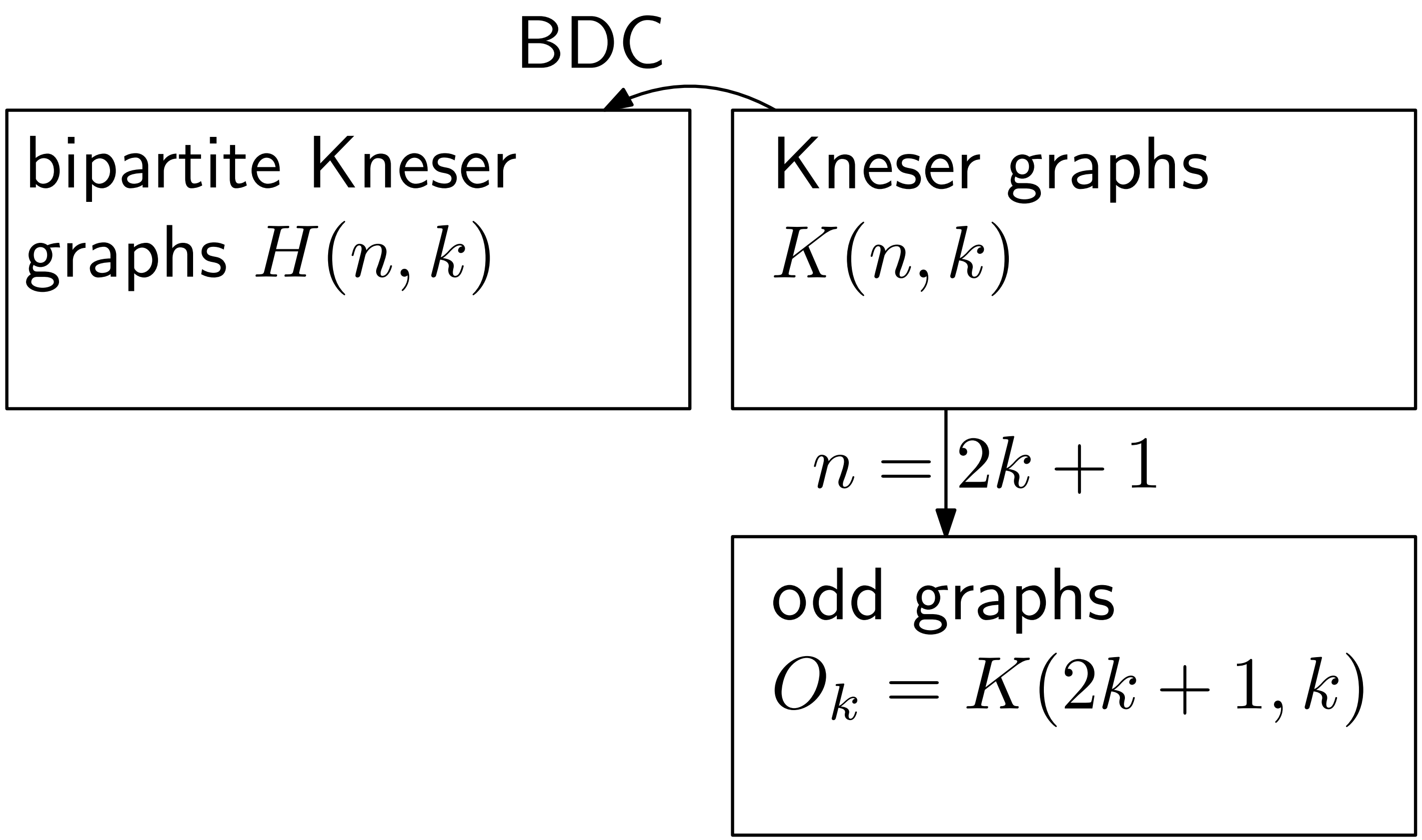

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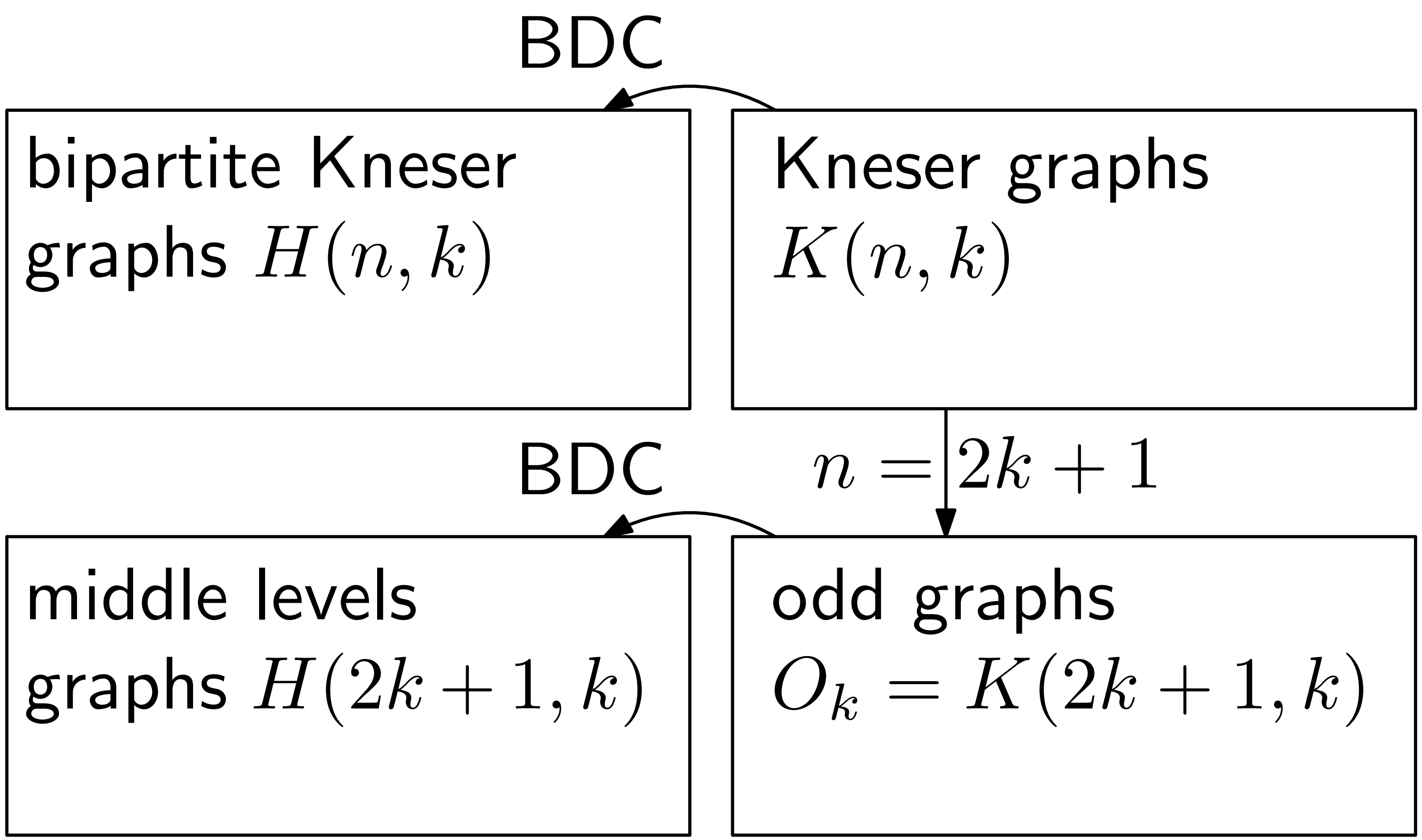

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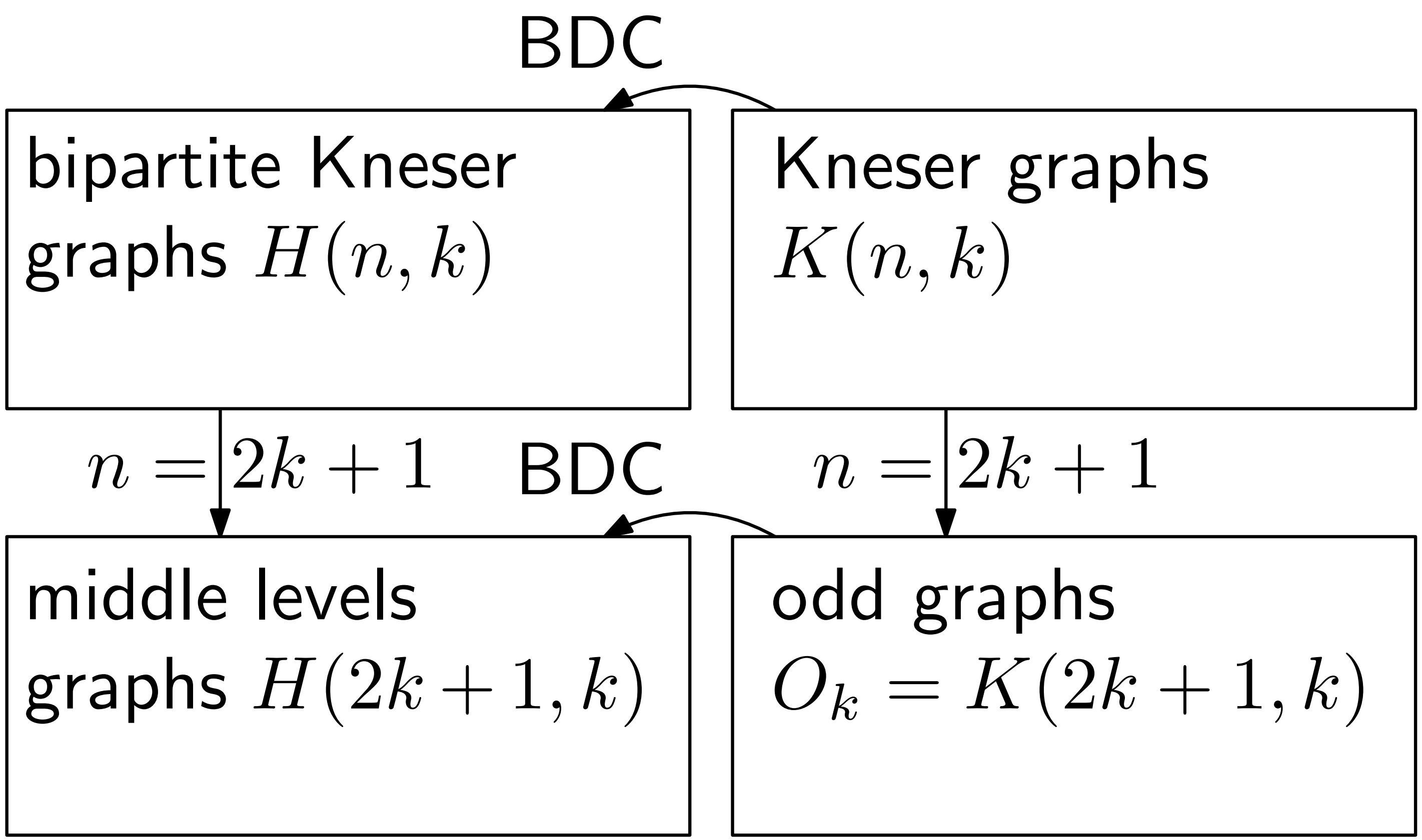

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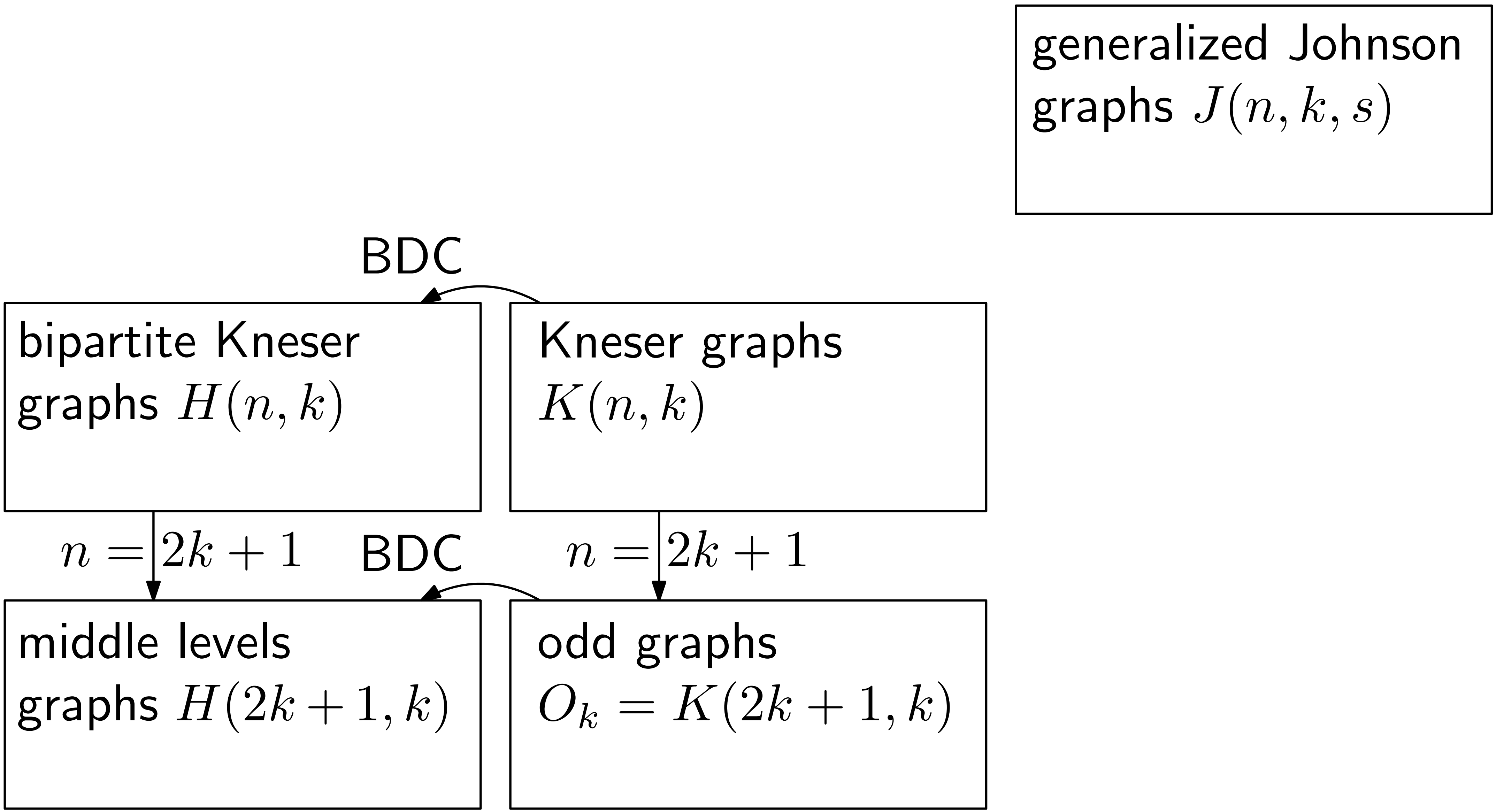

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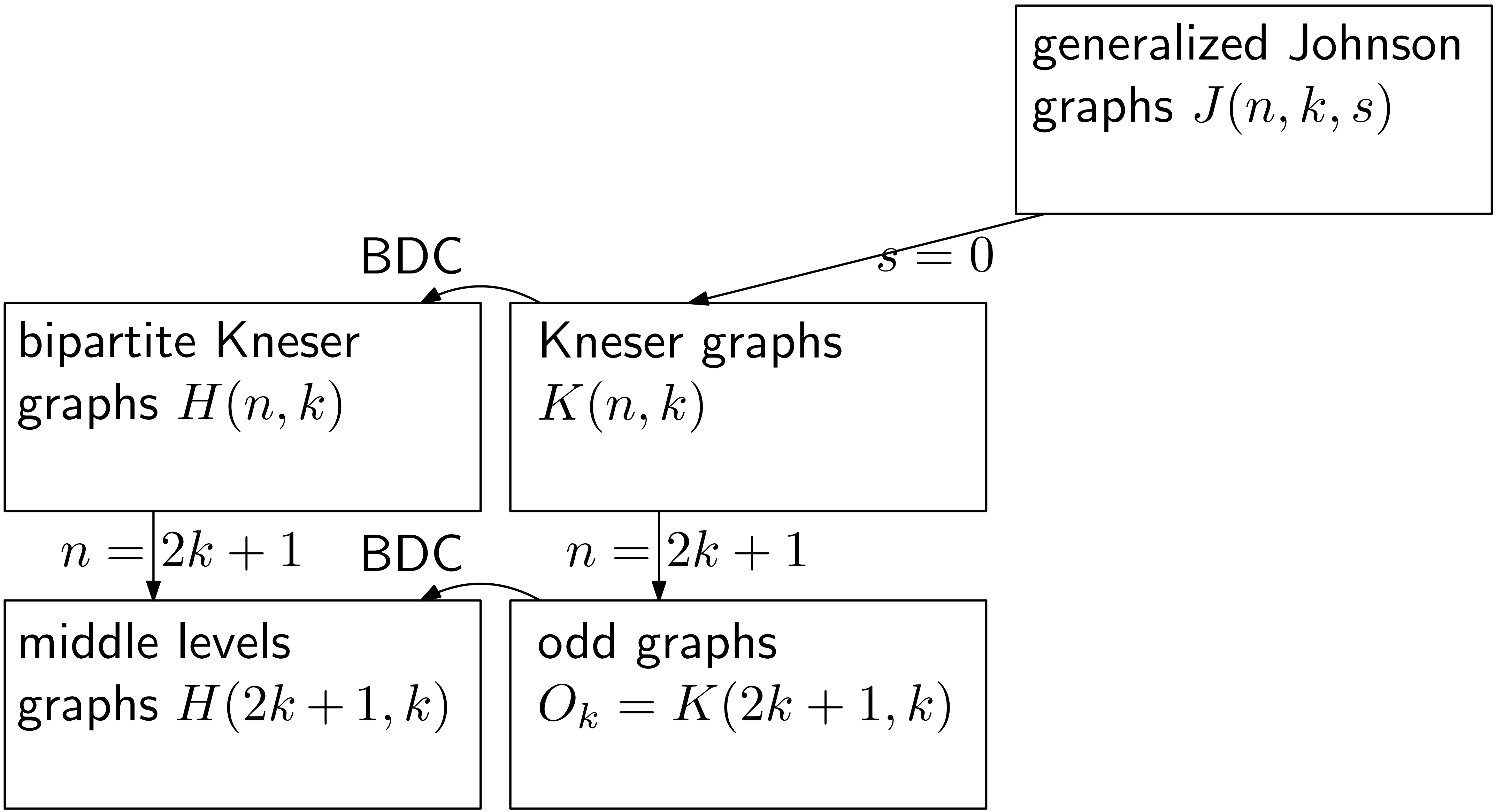

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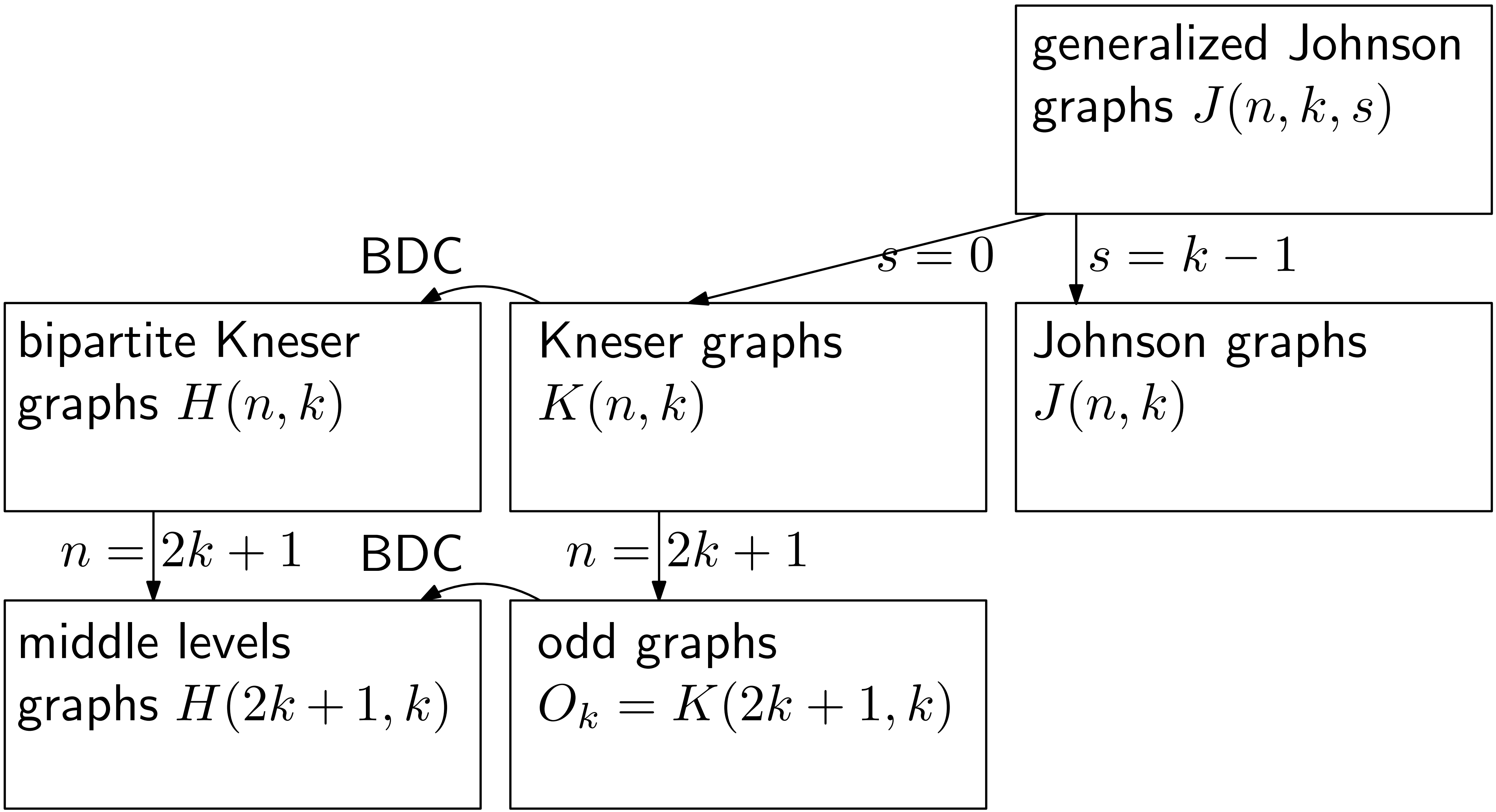

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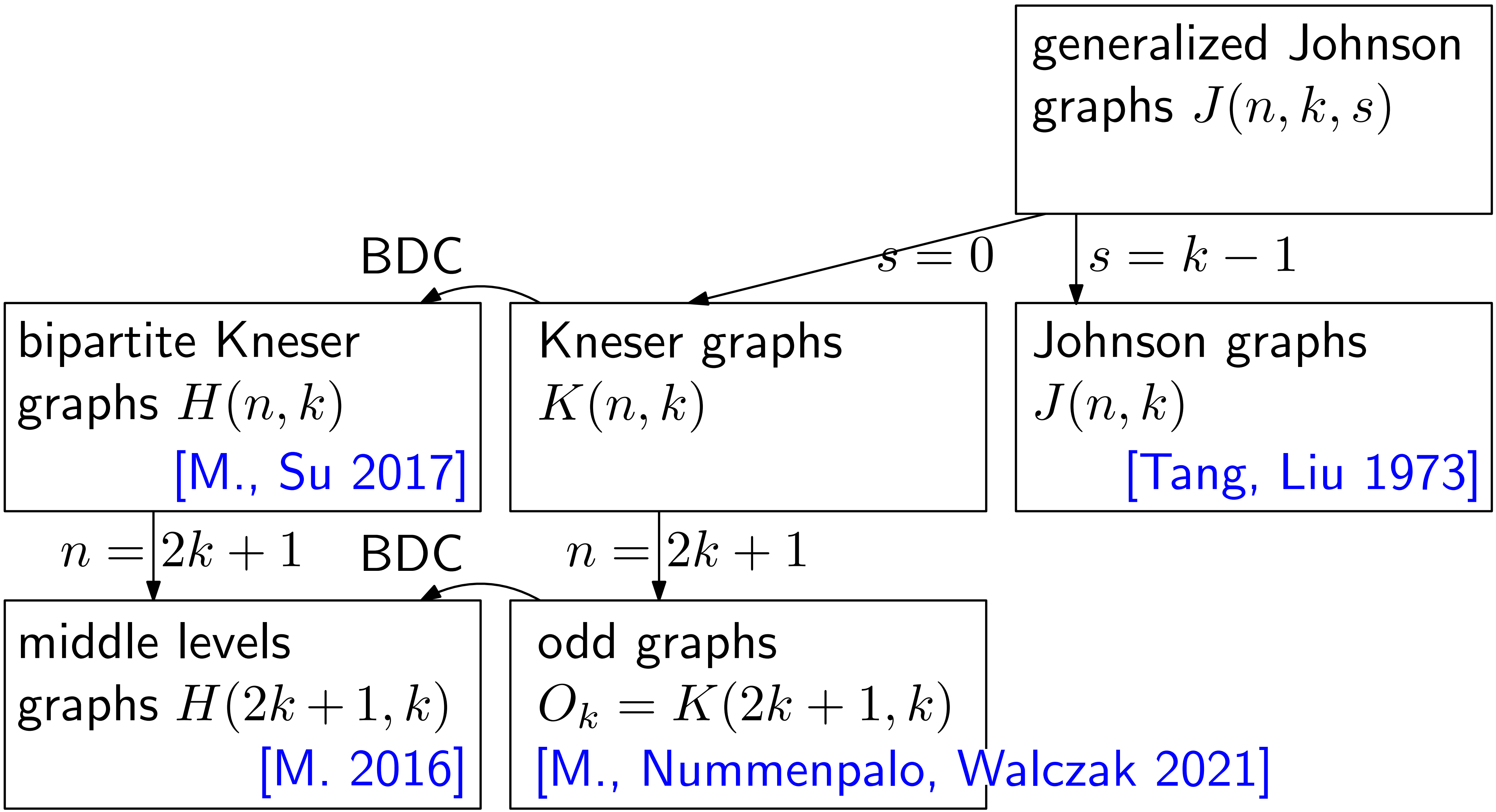

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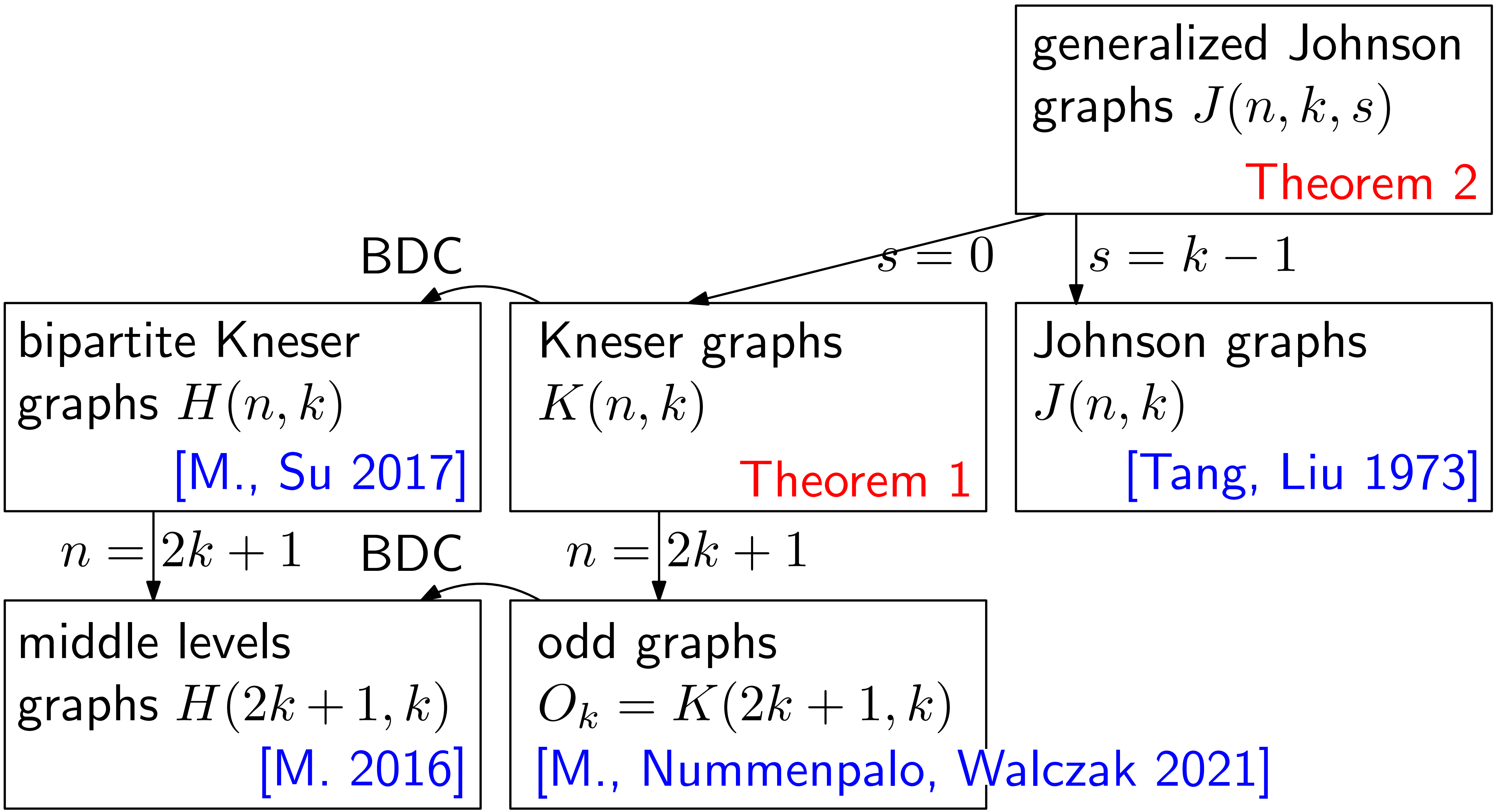

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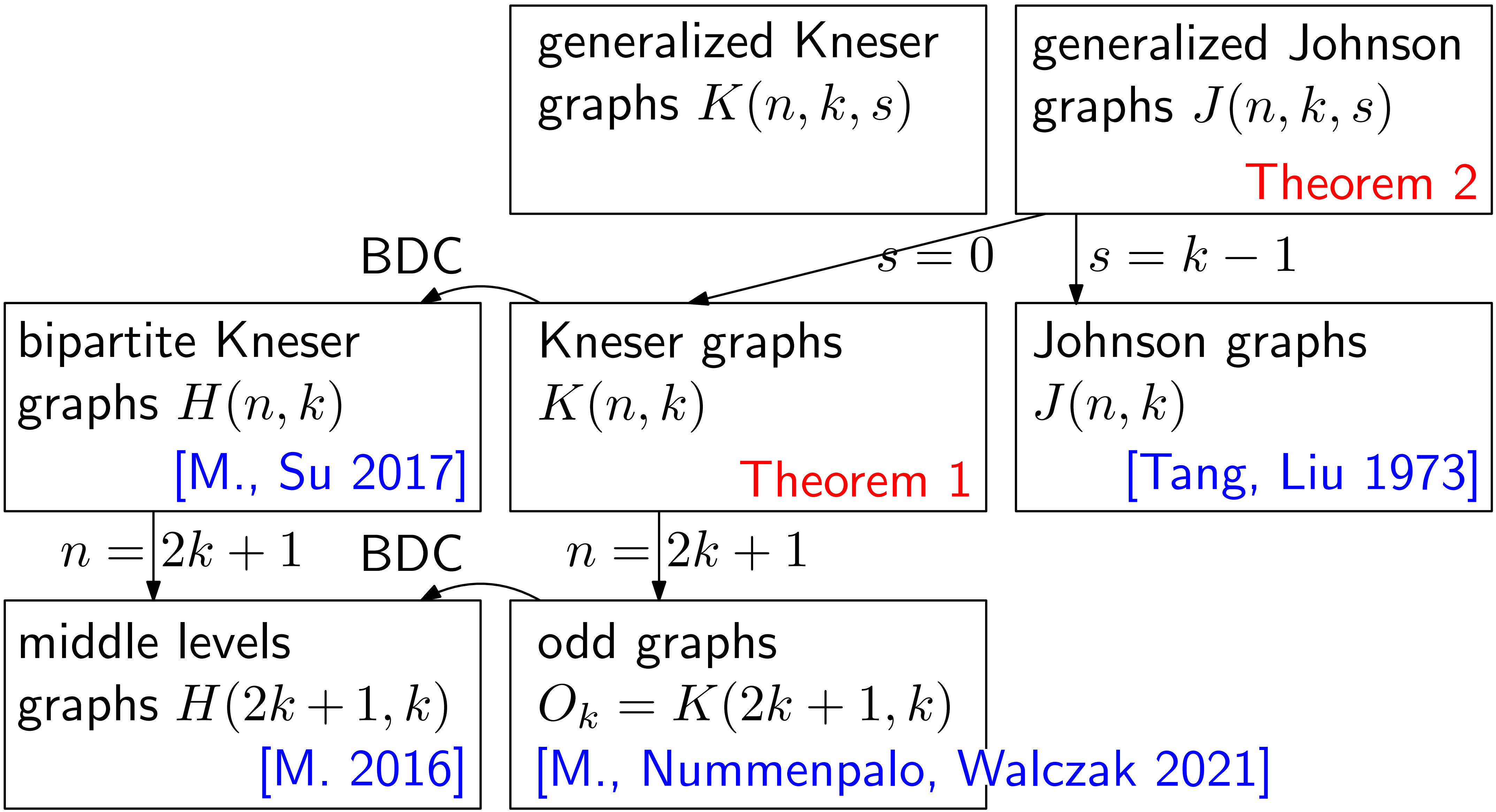

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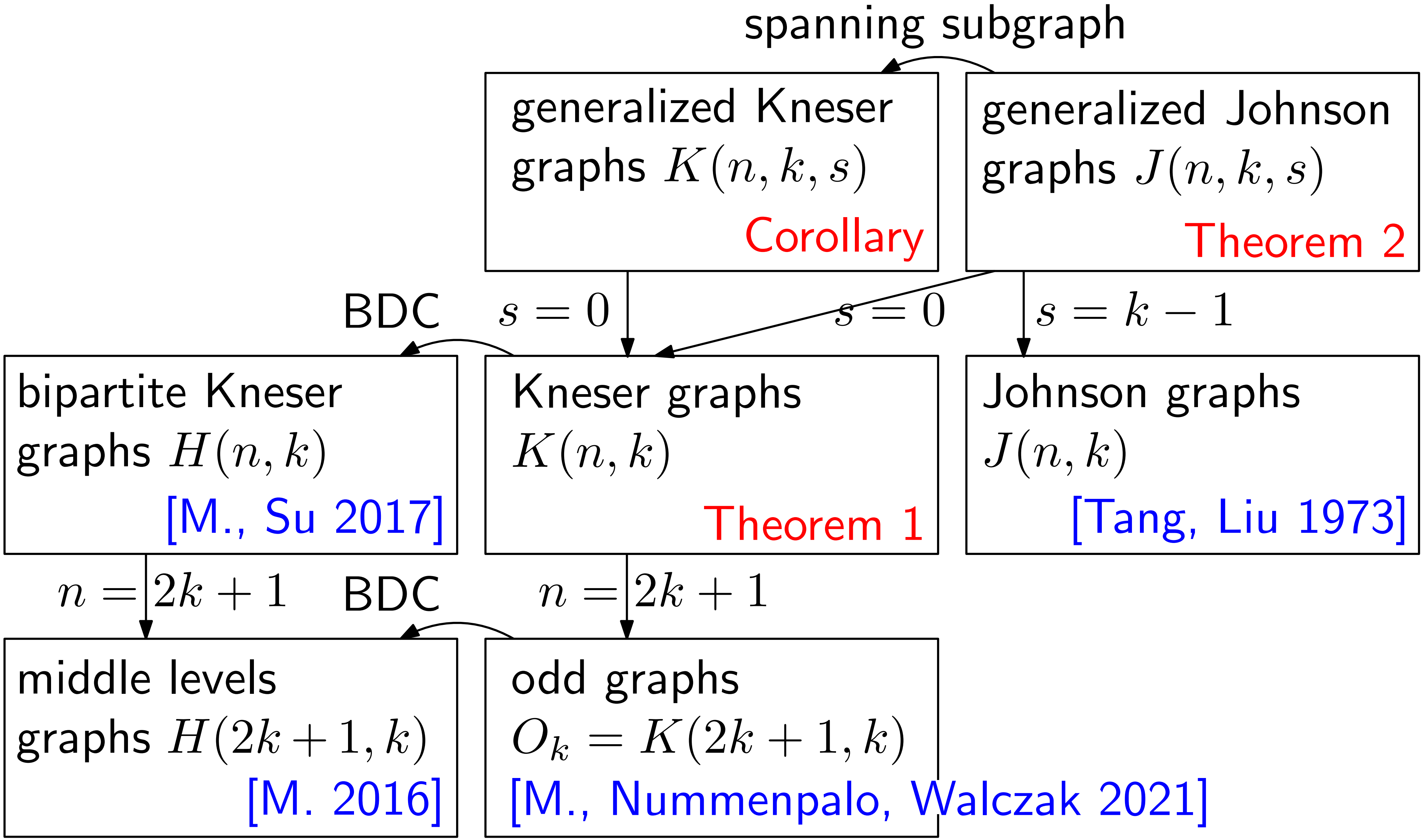

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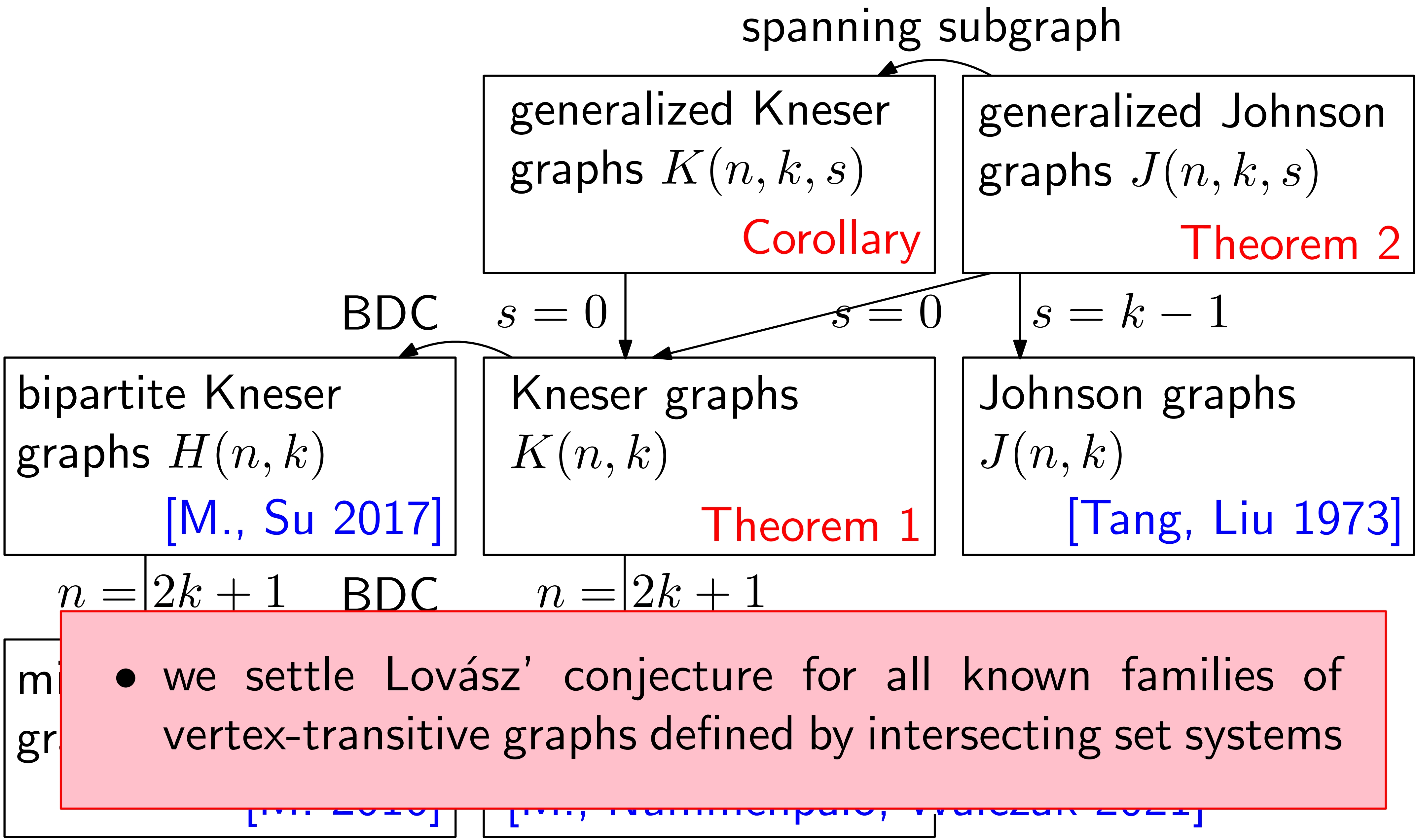

- we settle Lovász' conjecture for all known families of vertex-transitive graphs defined by intersecting set systems

Proof outline

- two sparsest cases $n = 2k + 1$ and $n = 2k + 2$ settled by [M., Nummenpalo, Walczak 2021]+[Johnson 2011]

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  - model cycles by kinetic system of interacting particles
  - reminiscent of the gliders in Conway's game of Life

Proof outline

- two sparsest cases $n = 2k + 1$ and $n = 2k + 2$ settled by [M., Nummenpalo, Walczak 2021]+[Johnson 2011]
- new proof assumes $n \geq 2k + 3$
  1. construct a cycle factor (works for $n \geq 2k + 1$)
  2. glue cycles together (needs $n \geq 2k + 3$)
- requires analyzing the cycles
  - model cycles by kinetic system of interacting particles
  - reminiscent of the gliders in Conway's game of Life
  - main technical innovation

Cycle factor

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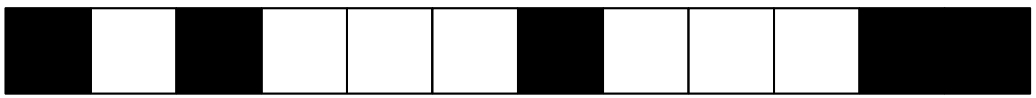
1 0 1 0 0 0 1 0 0 0 1 1



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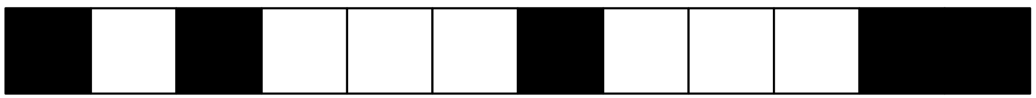

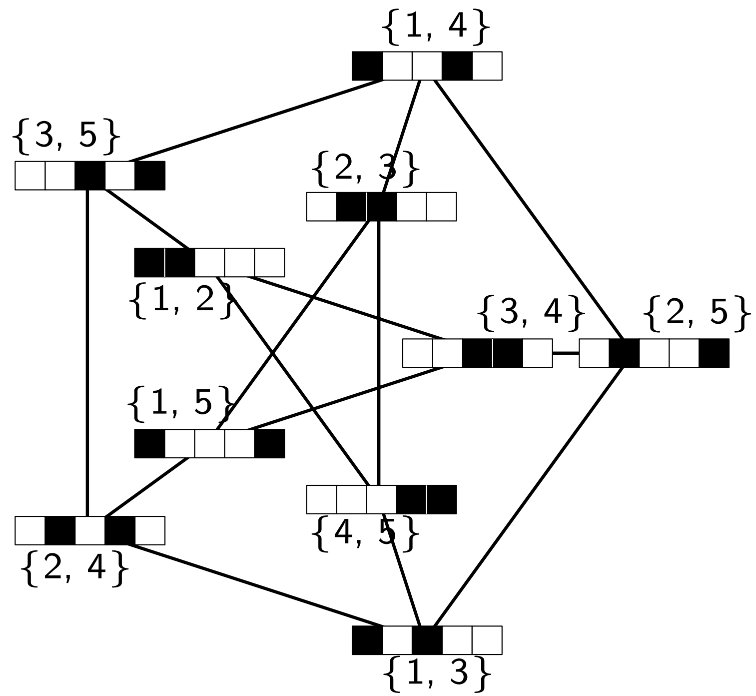

Cycle factor

- parenthesis matching with 1=[and 0=] (cyclically)

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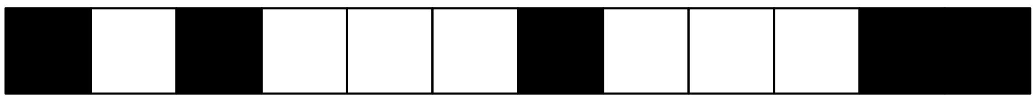
1 0 1 0 0 0 1 0 0 0 1 1



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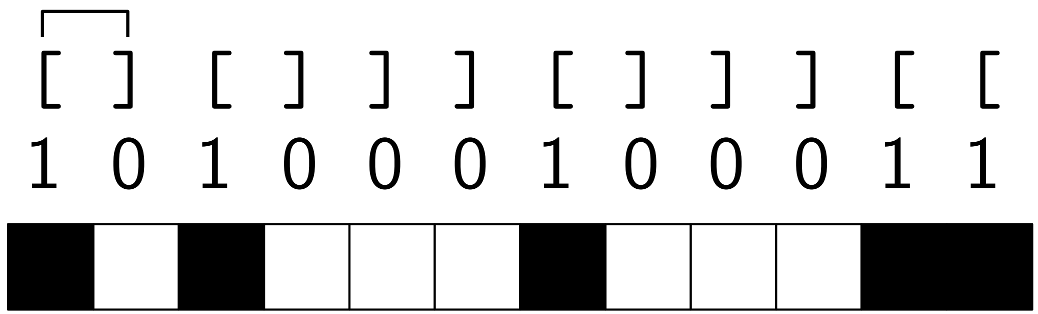

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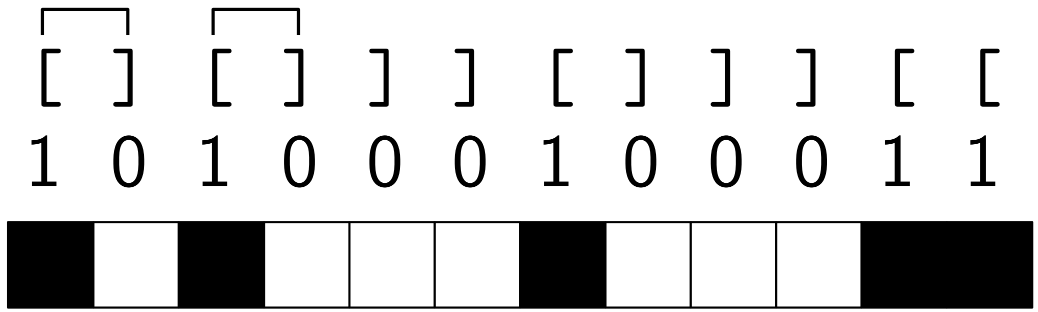

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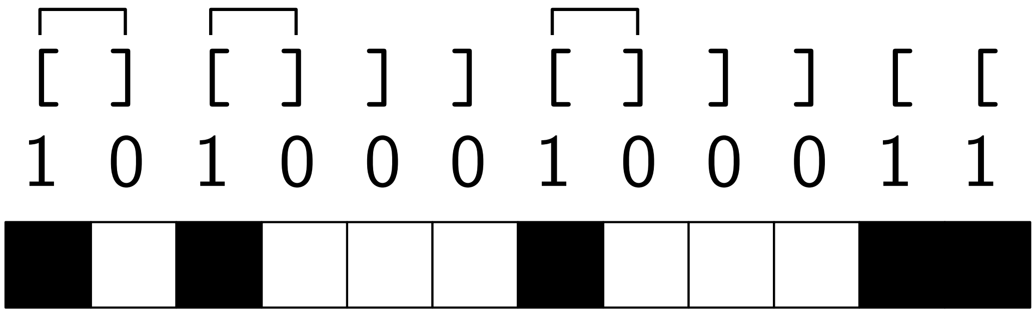

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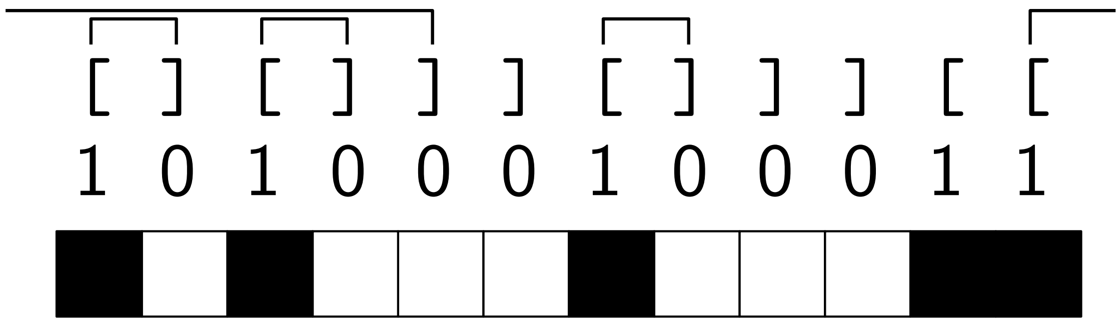

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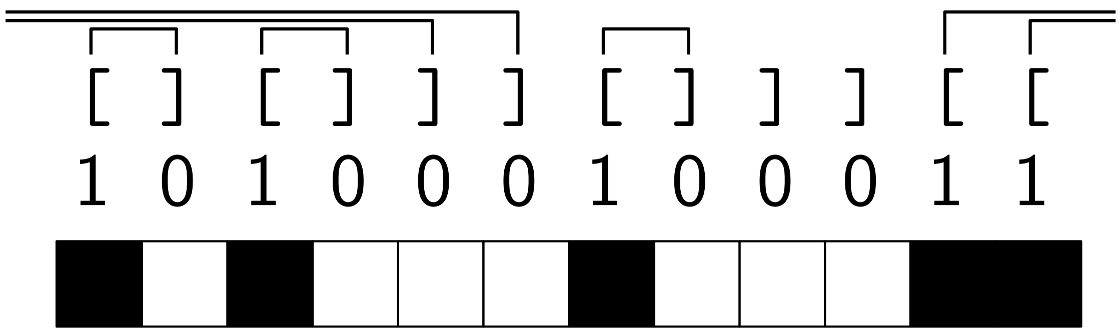

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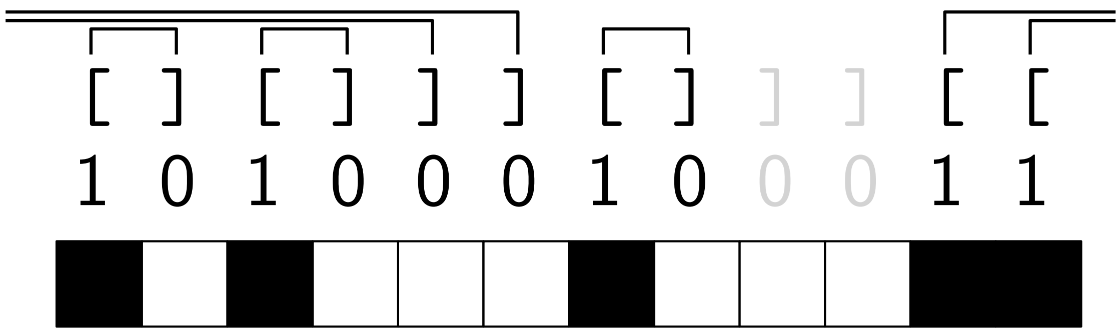
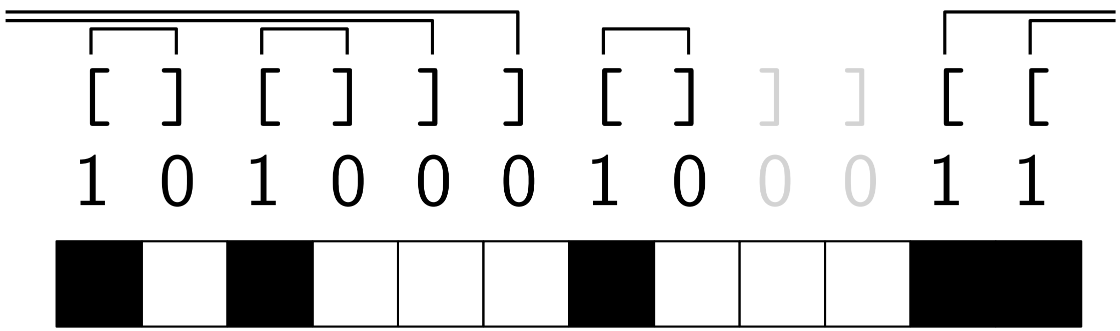


Cycle factor

- parenthesis matching with 1=[and 0=] (cyclically)
- f: complement matched bits



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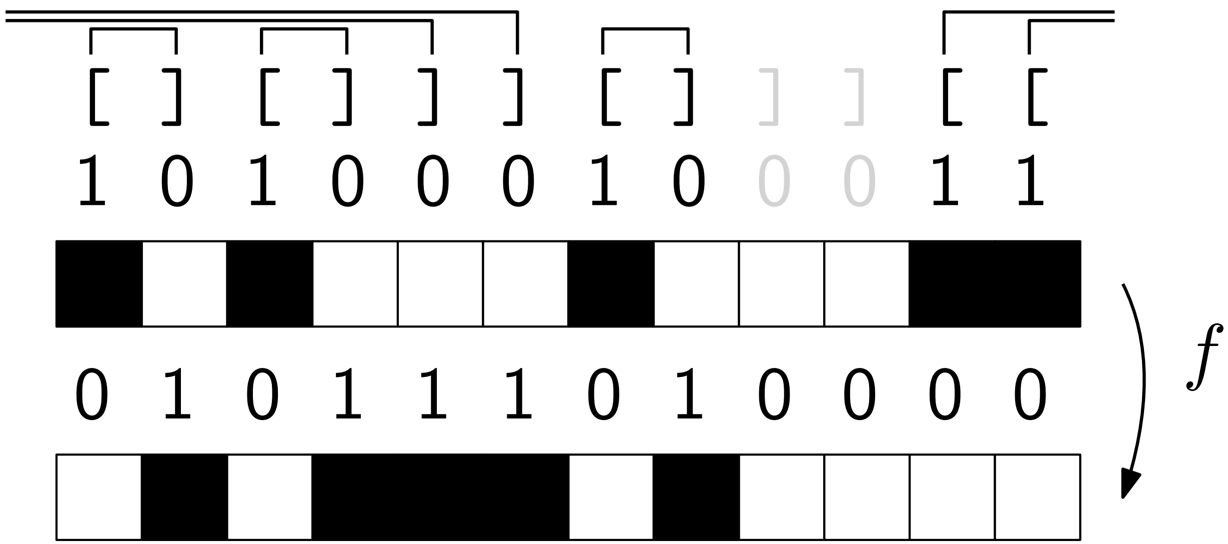

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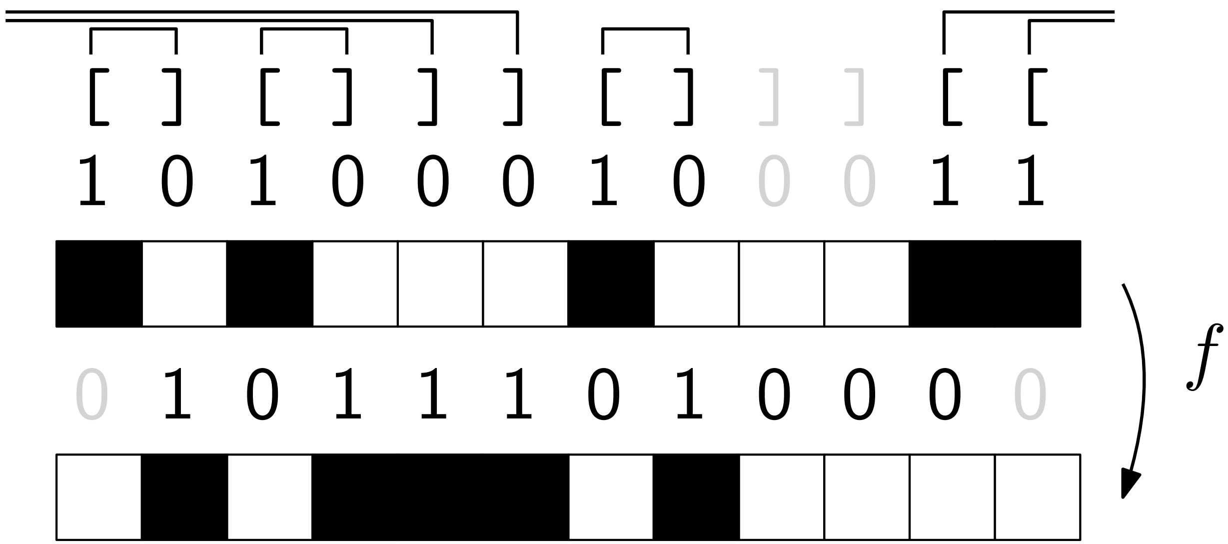

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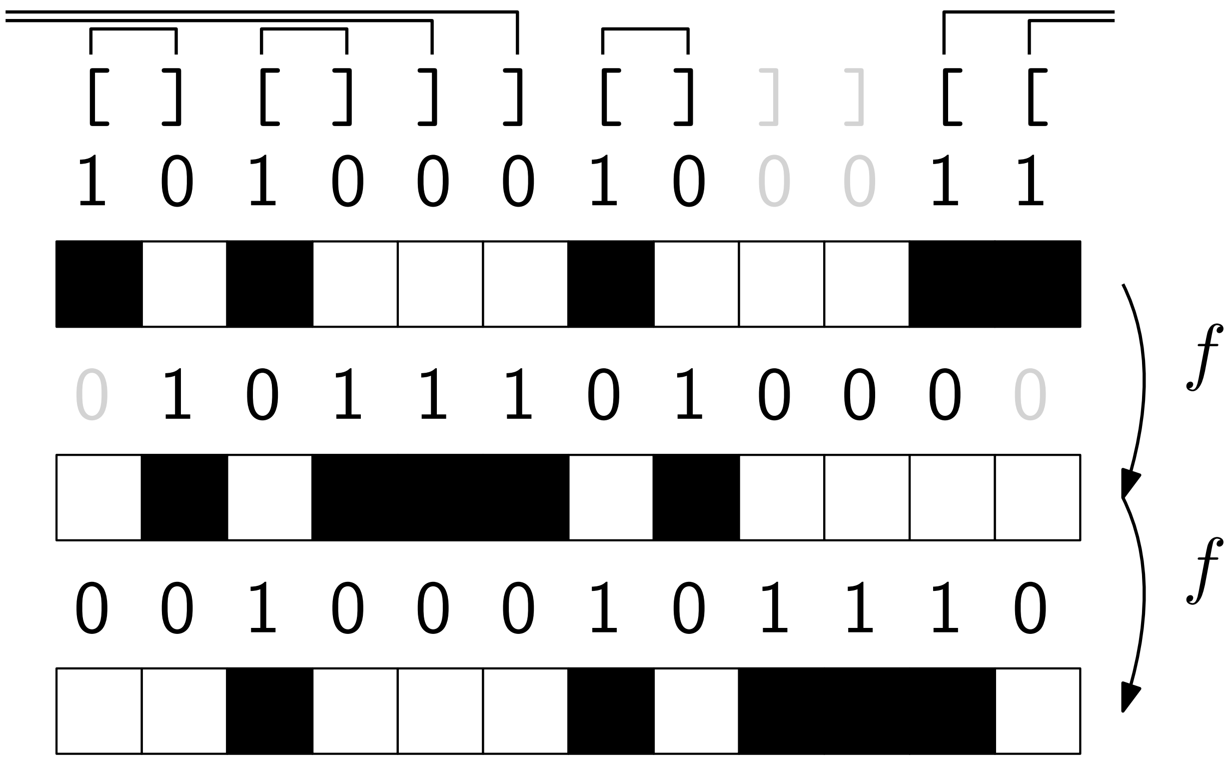

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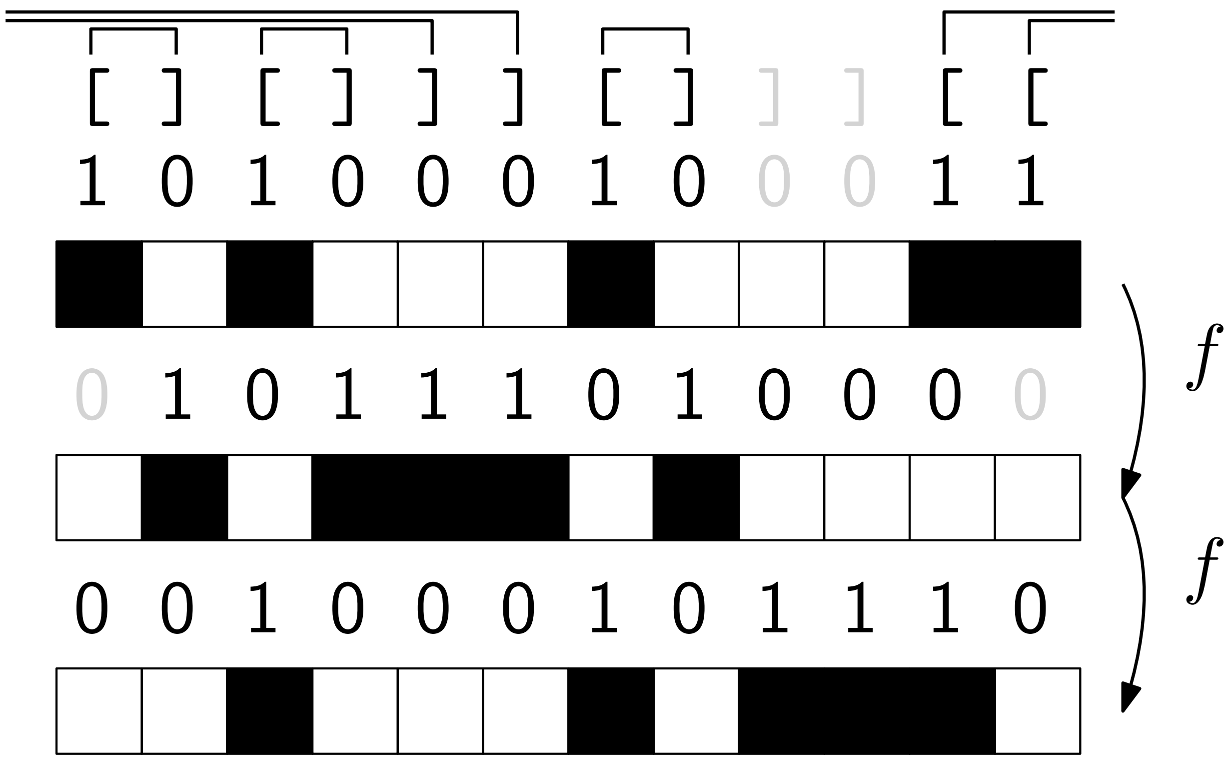
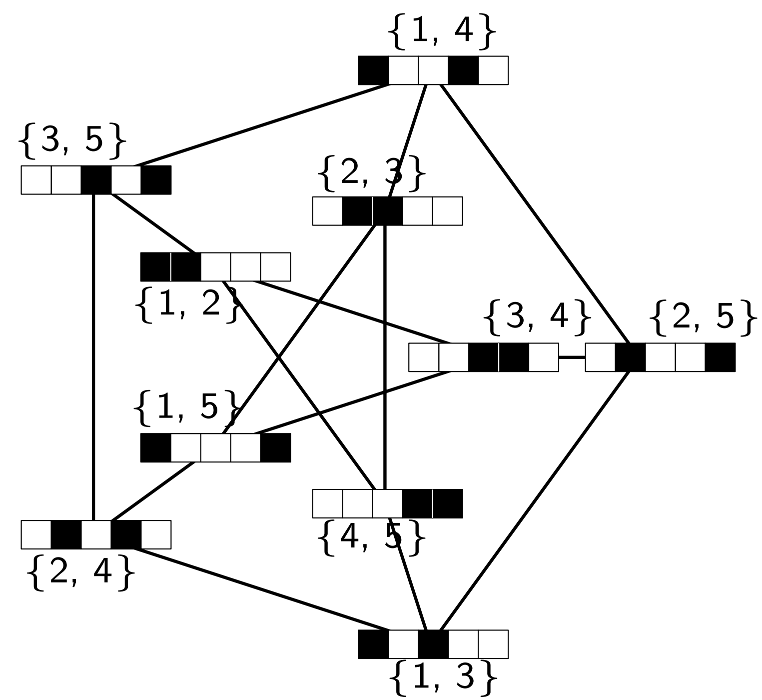


- f is invertible $\rightarrow$ partition of $K(n,k)$ into disjoint cycles

Cycle factor

- **Example:** $K(5,2)$



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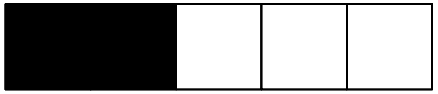

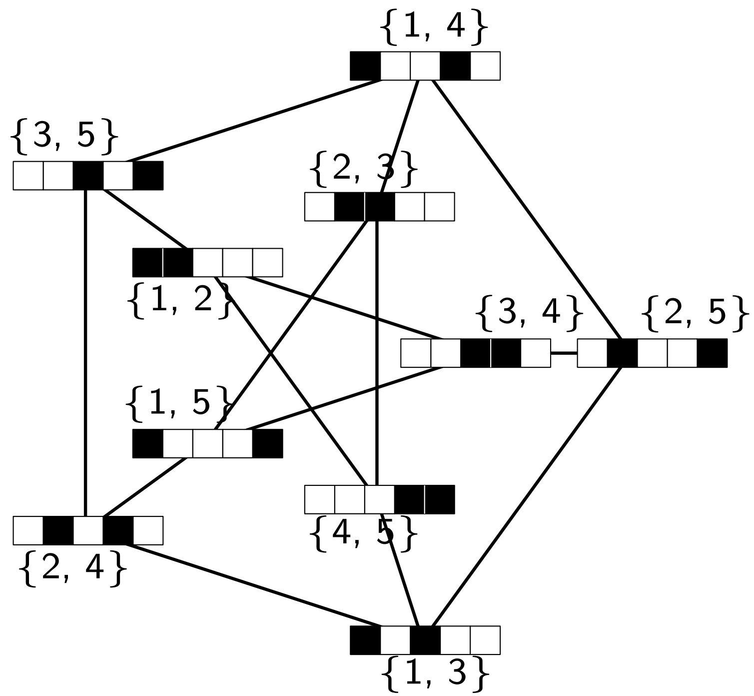

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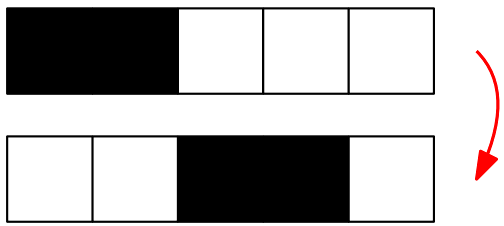

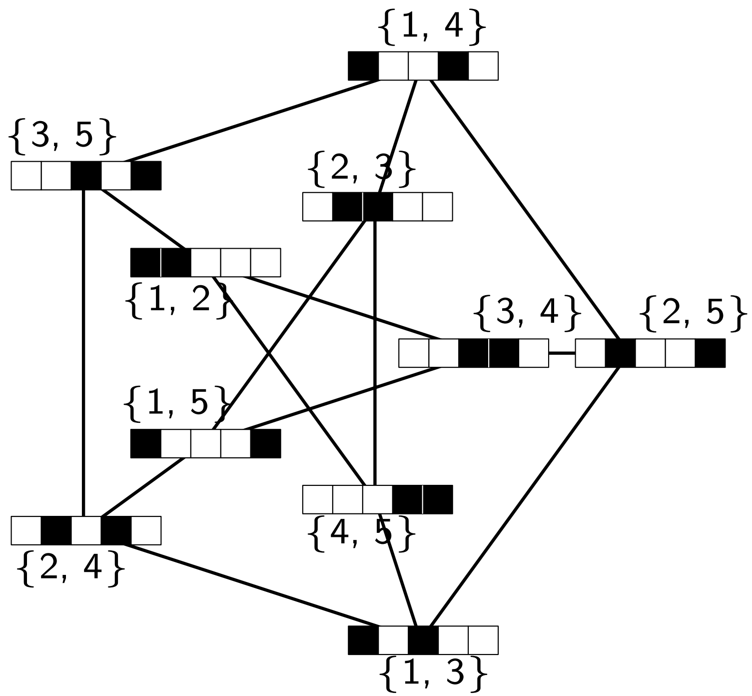

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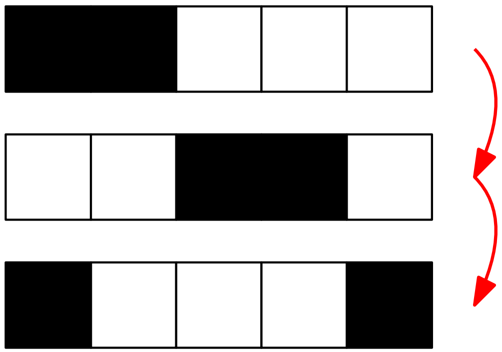

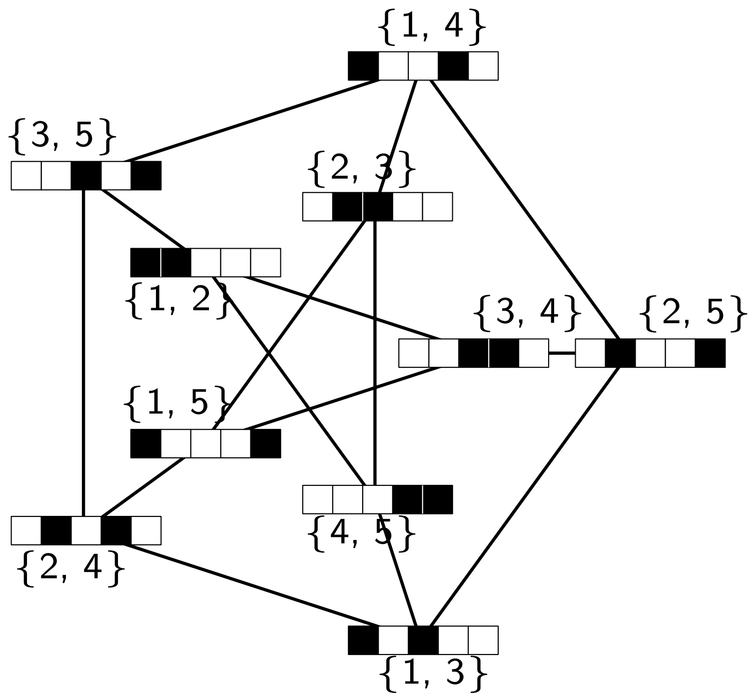

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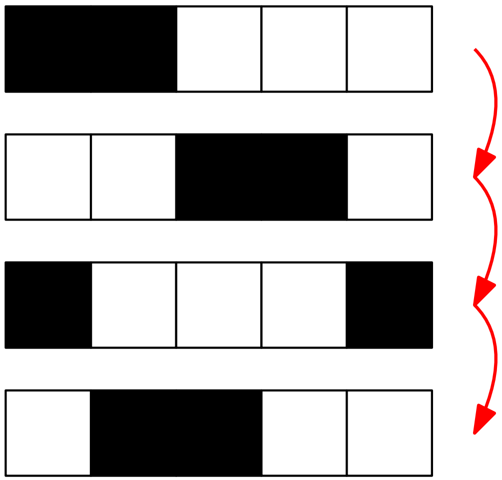

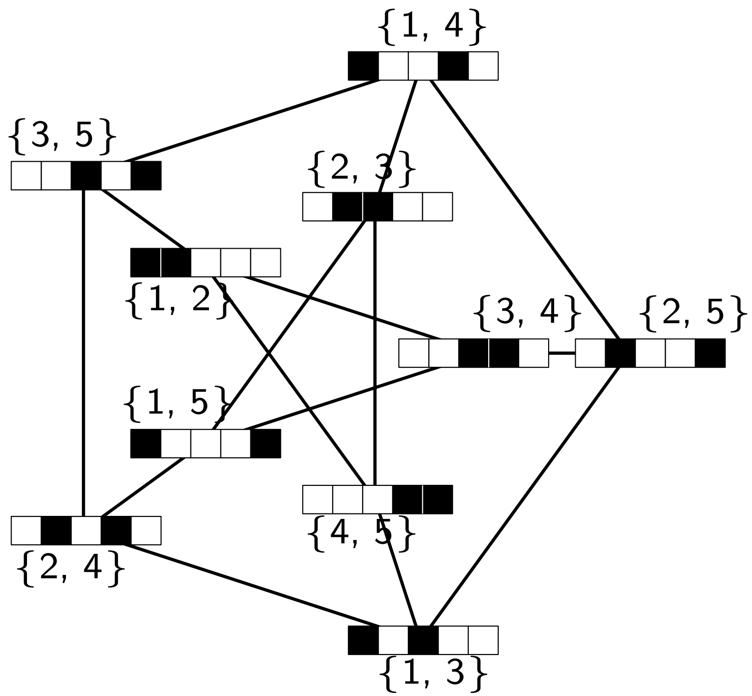

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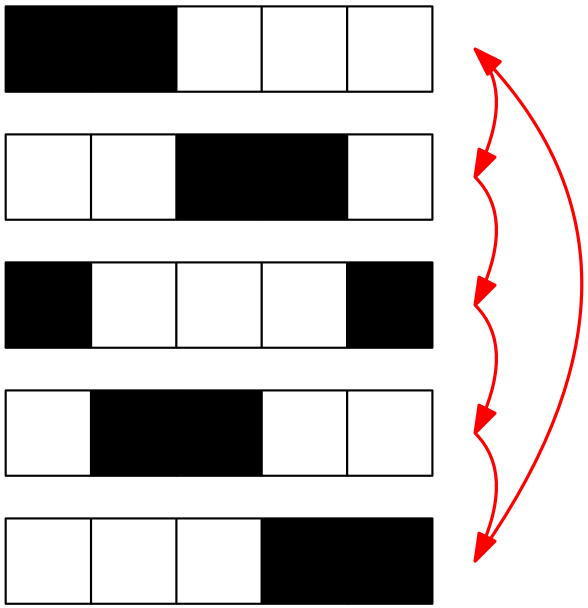

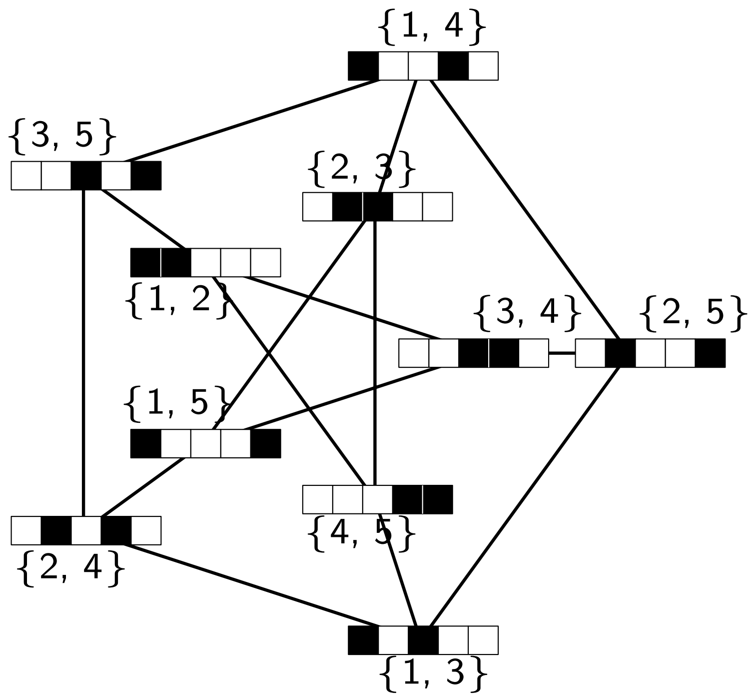

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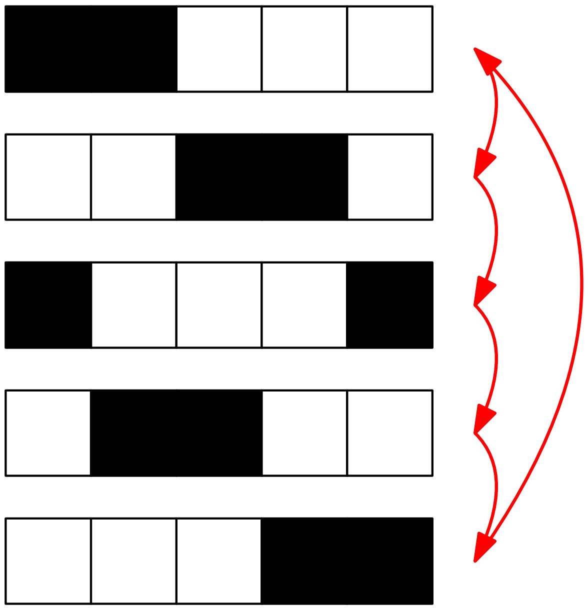

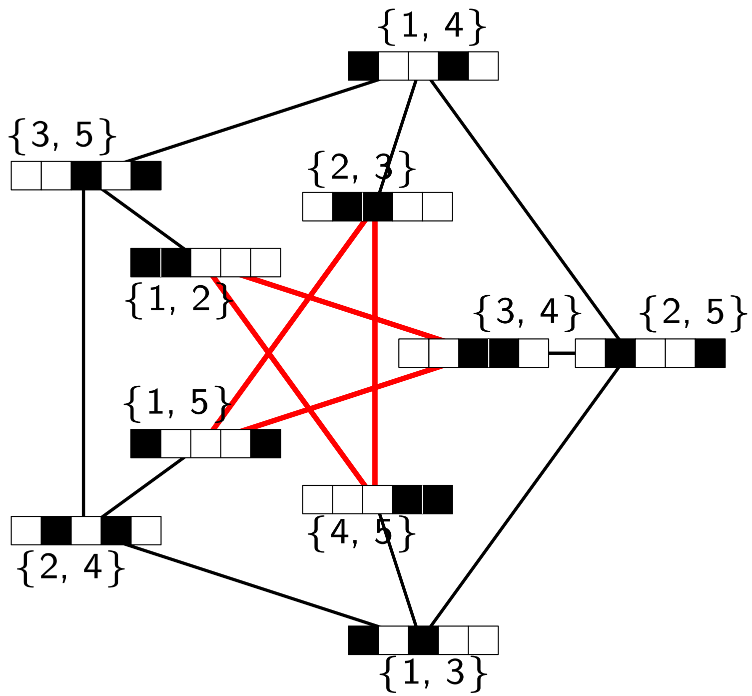

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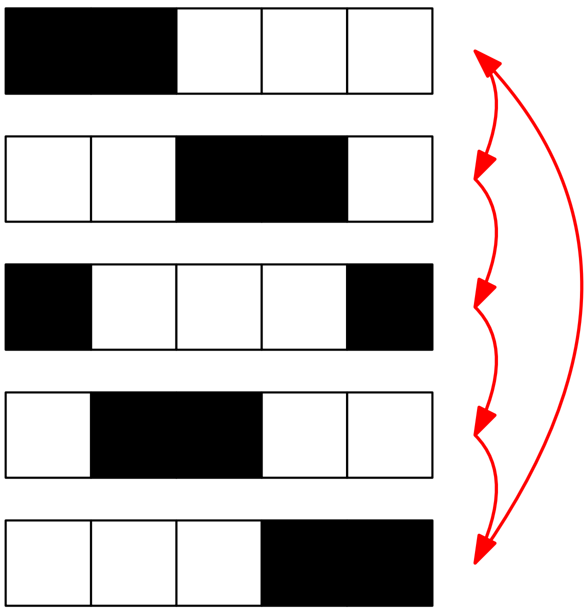

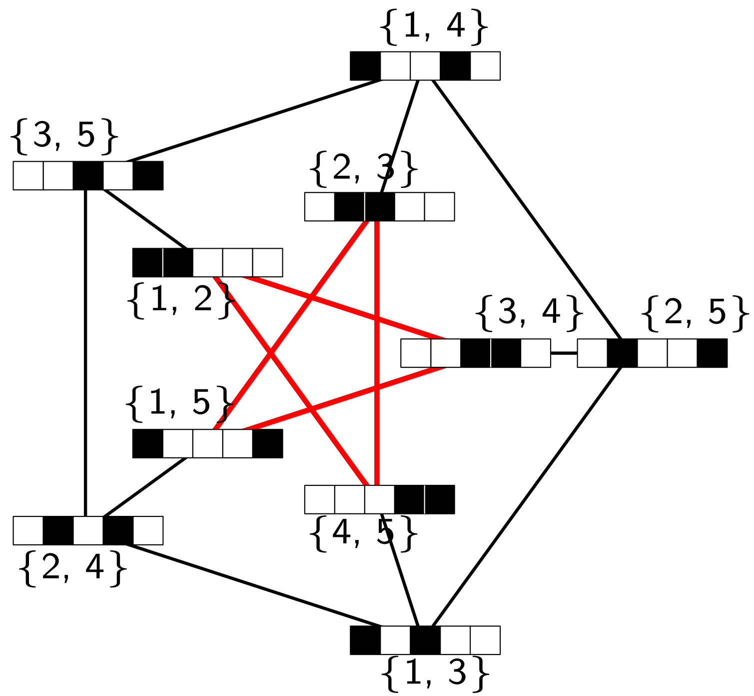

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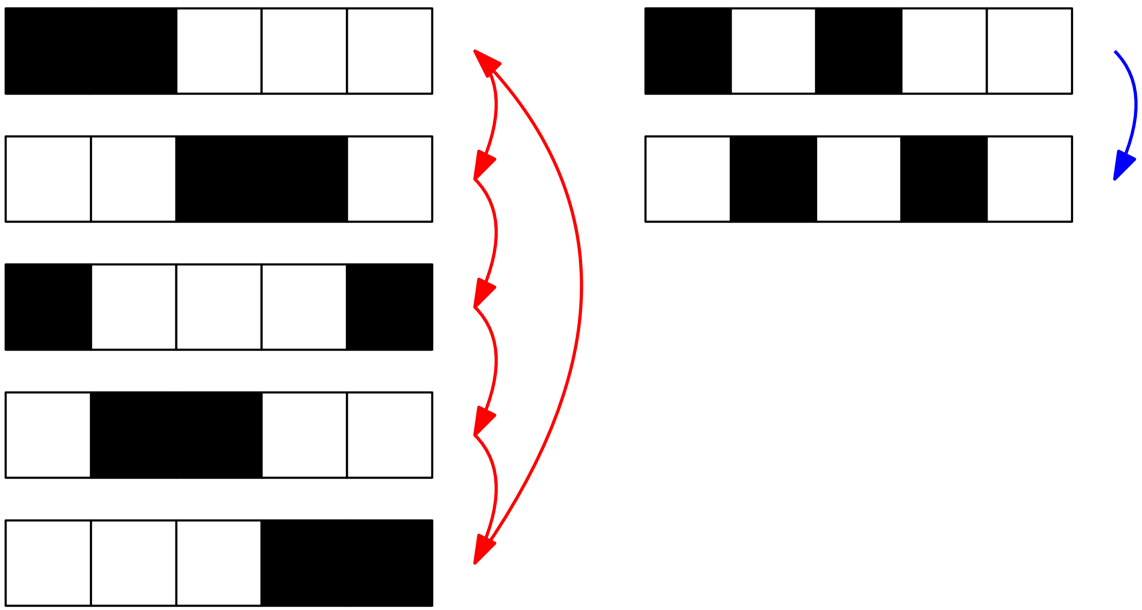

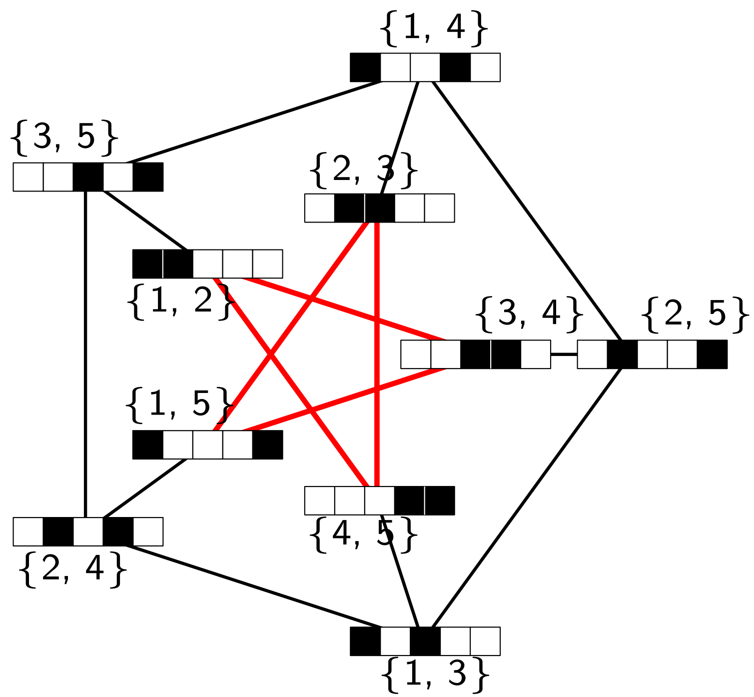

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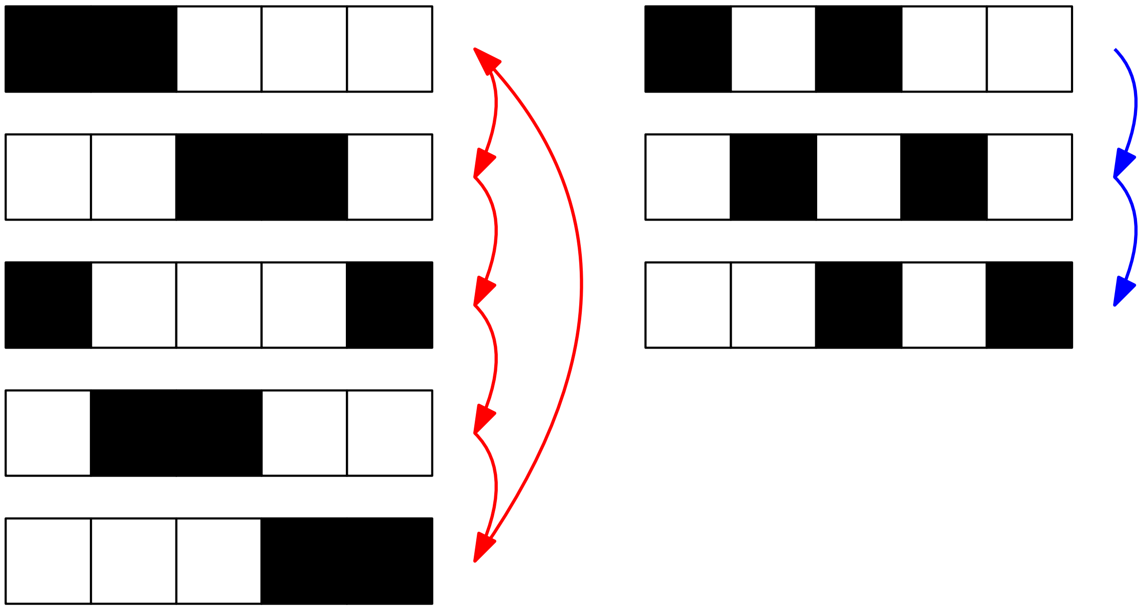

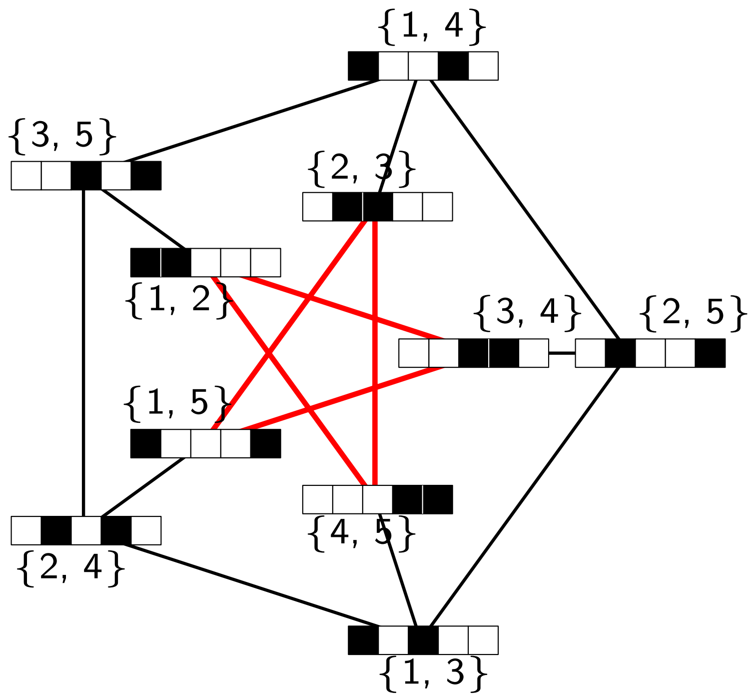

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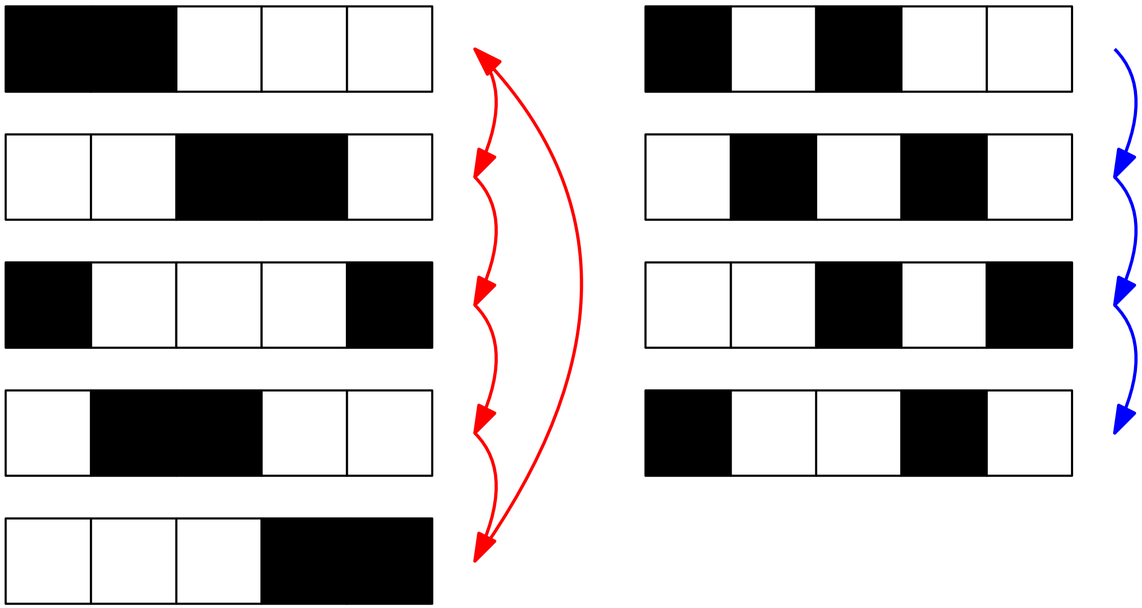

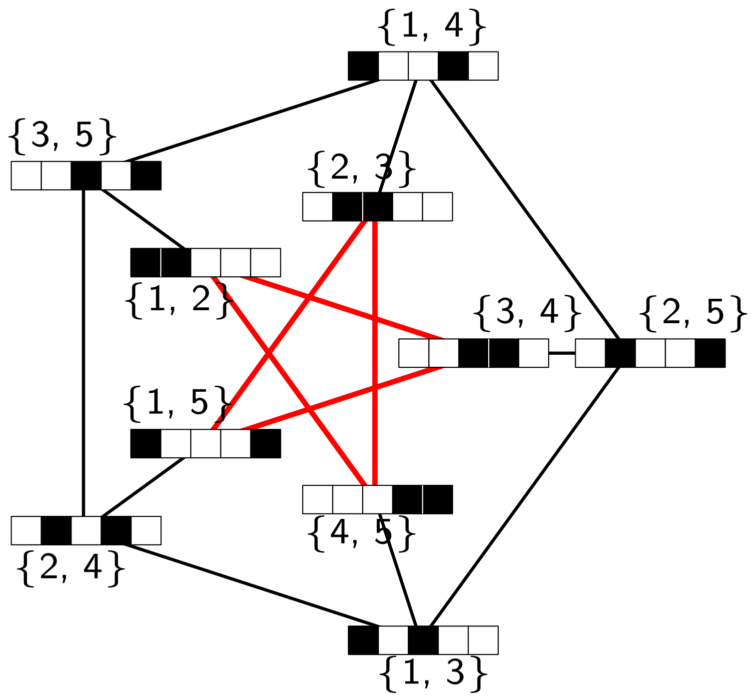

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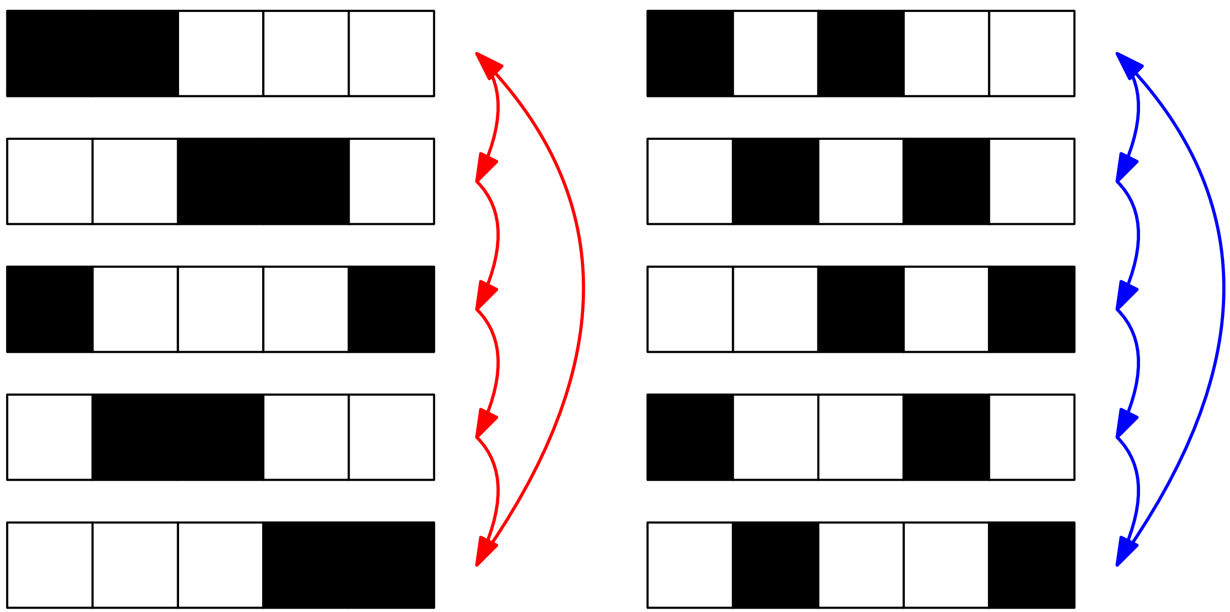

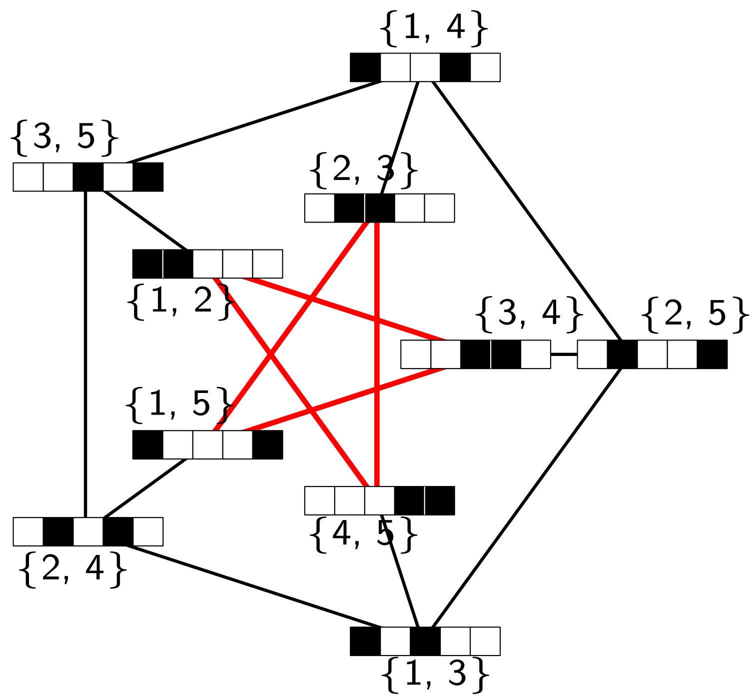

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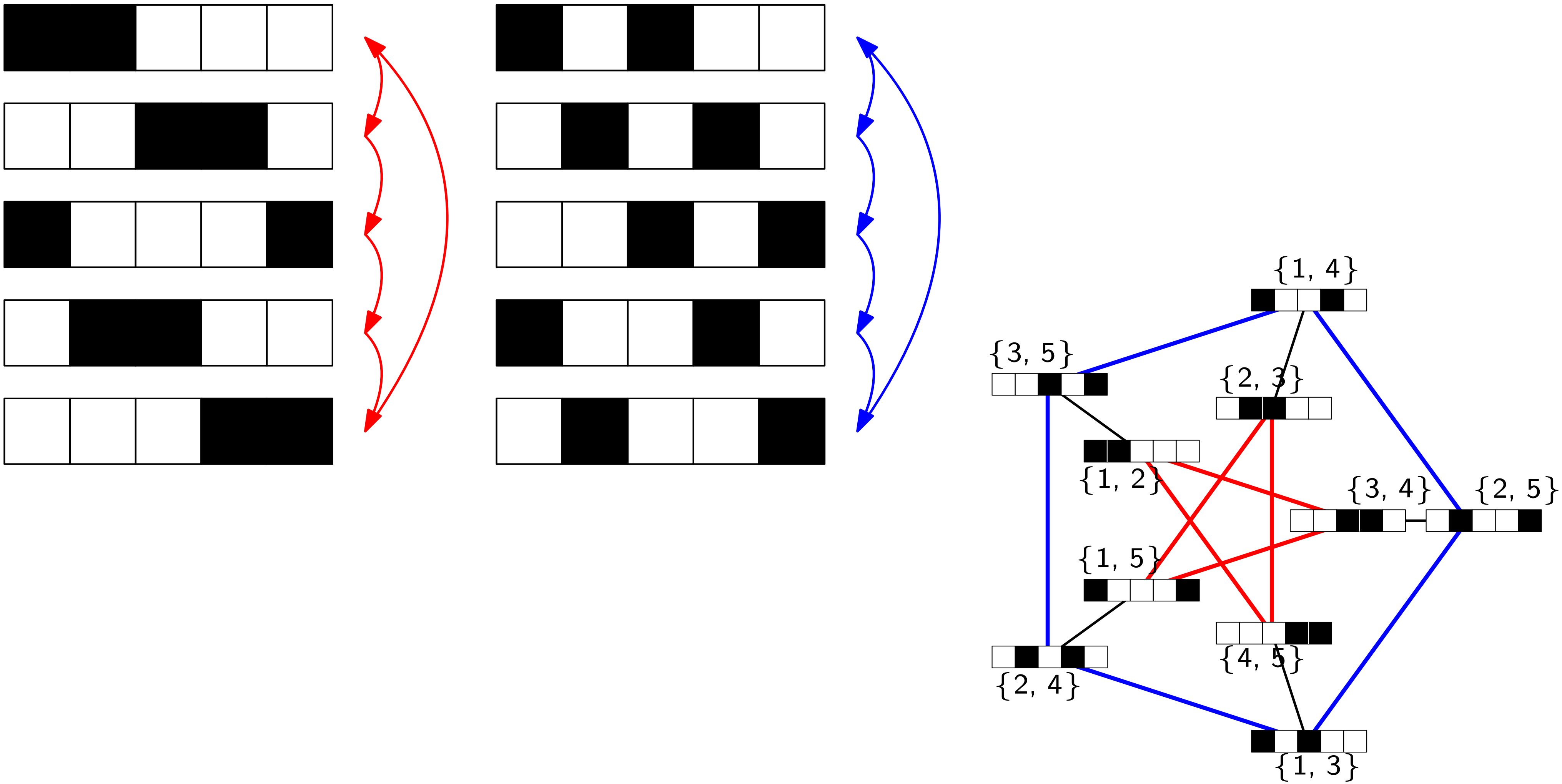
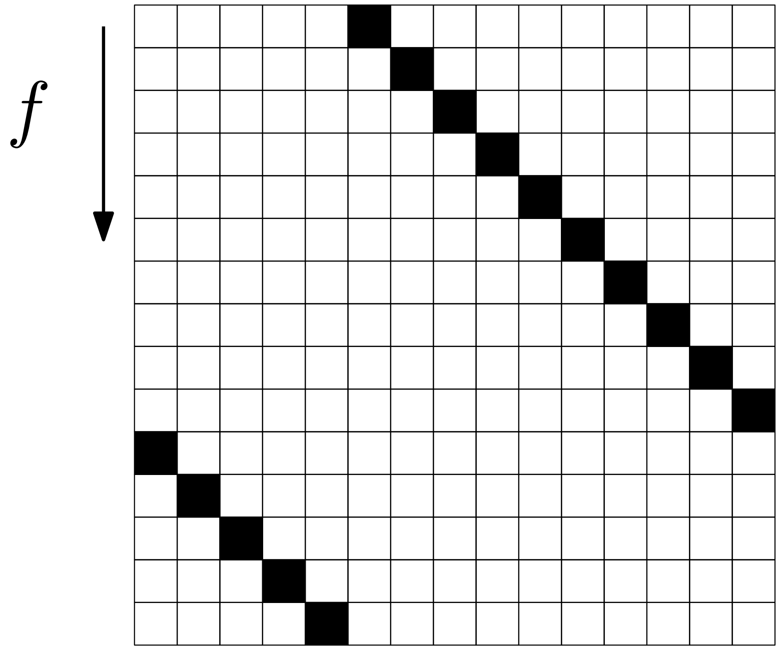


Analyzing the cycles

$(n, k) = (15, 1)$



Analyzing the cycles

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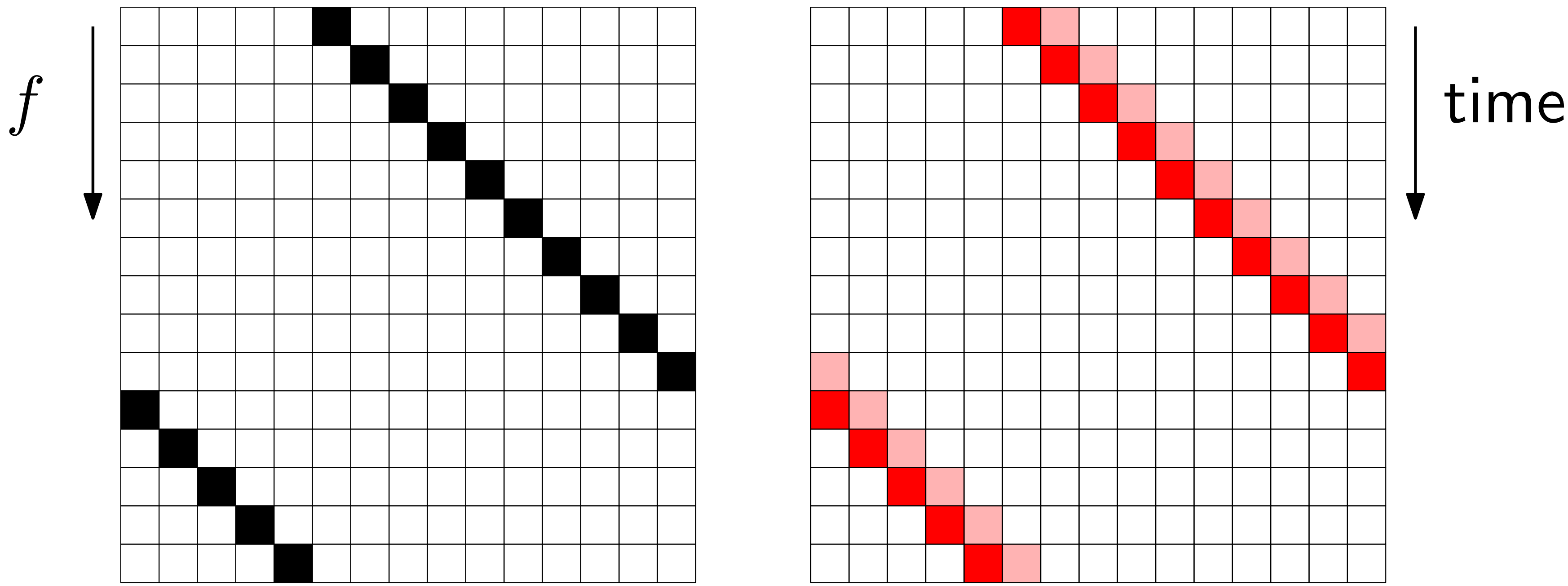

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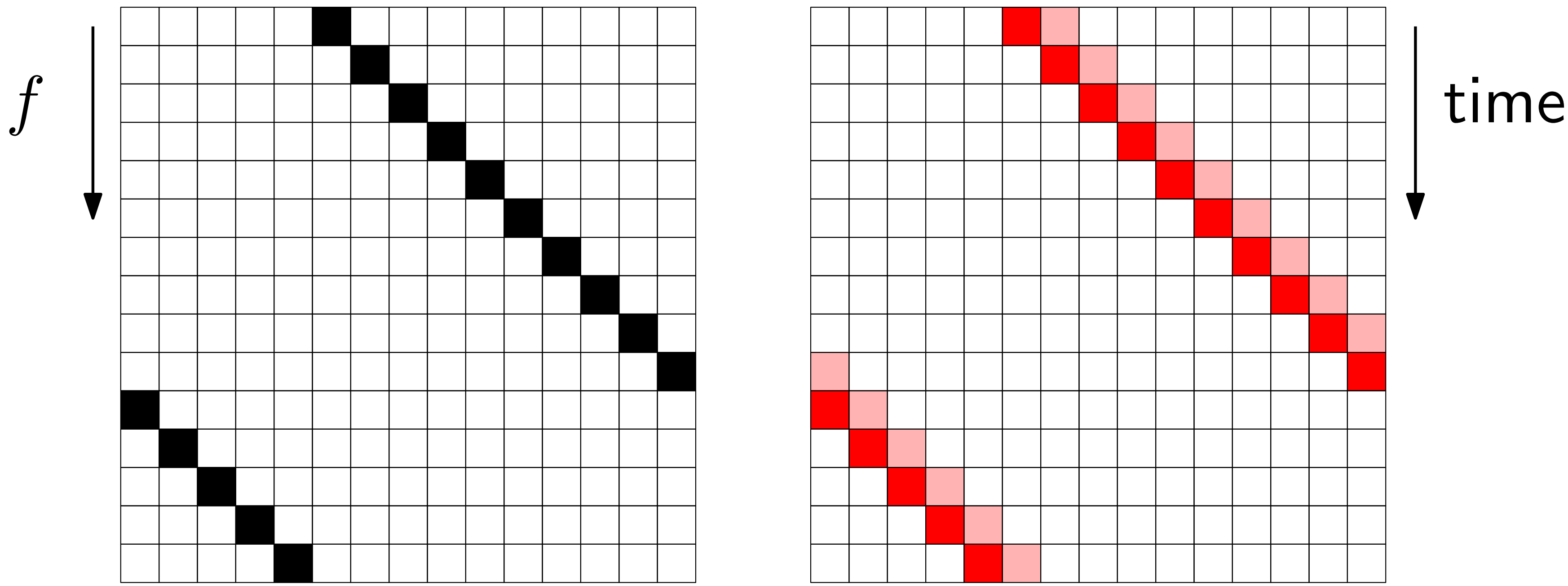

- Two matched bits form a glider
- Glider moves forward by 1 unit per step

Analyzing the cycles

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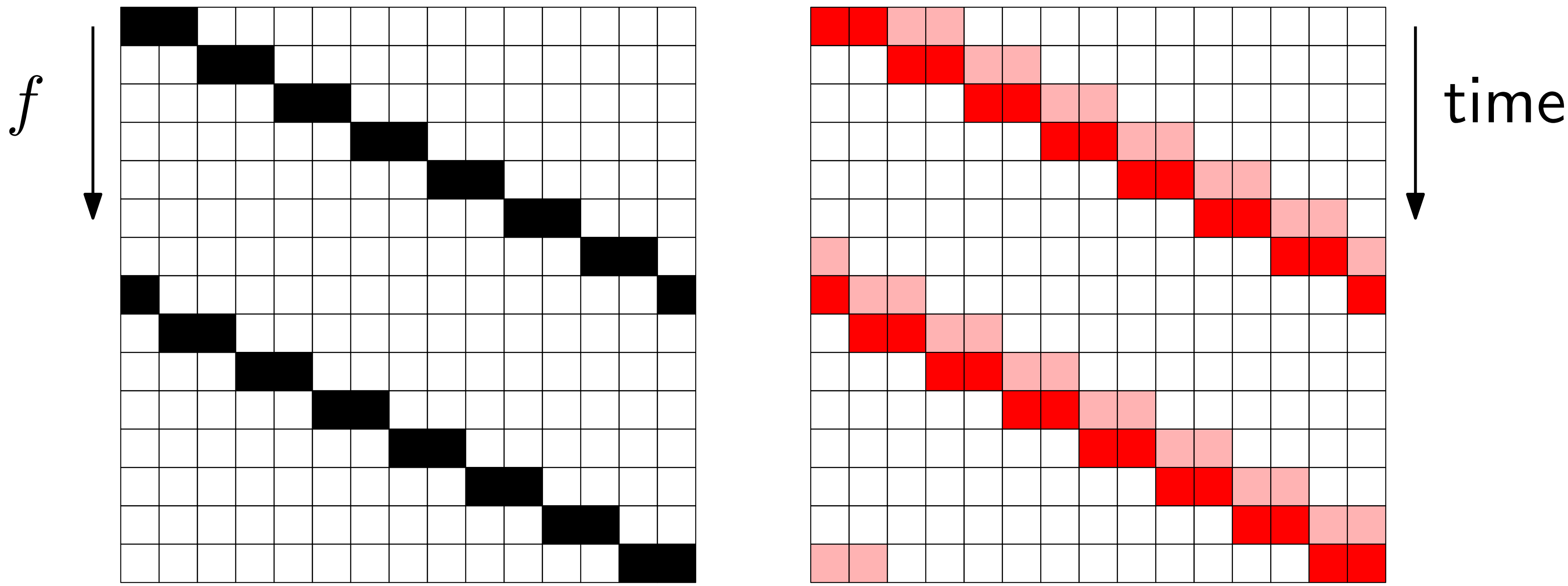
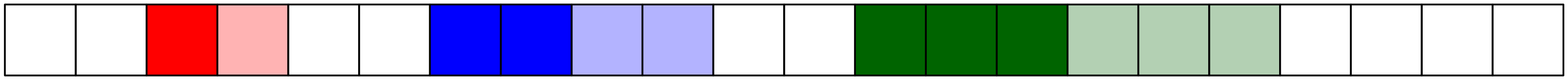


- Four matched bits form one glider
- Glider moves forward by 2 units per step

Gliders

- **glider** := set of matched 1s and 0s (same number of each)



Gliders

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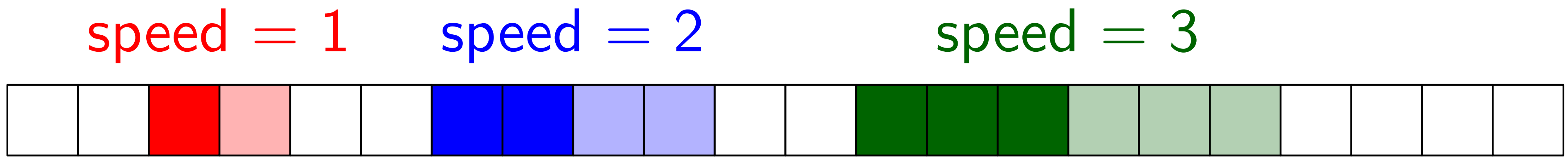
- **speed** := numbers of 1s = number of 0s



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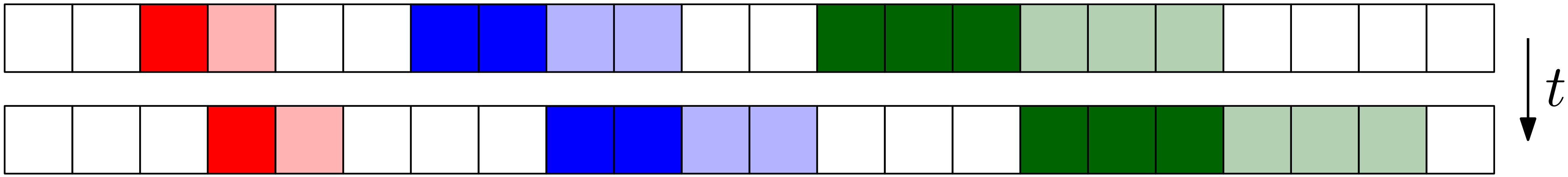
speed = 1 speed = 2 speed = 3



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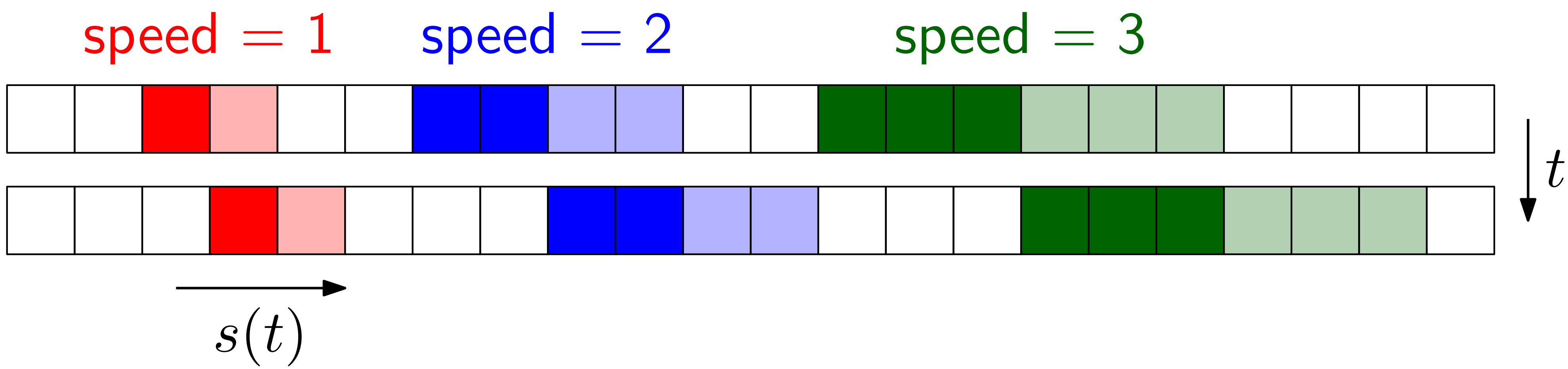

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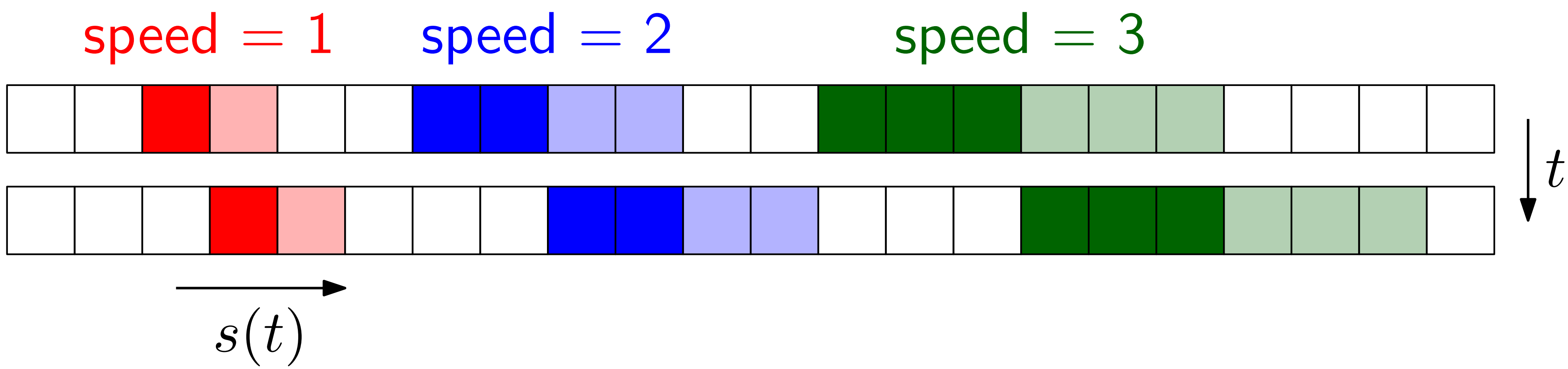

- Uniform equation of motion: $s(t) = v \cdot t + s(0)$

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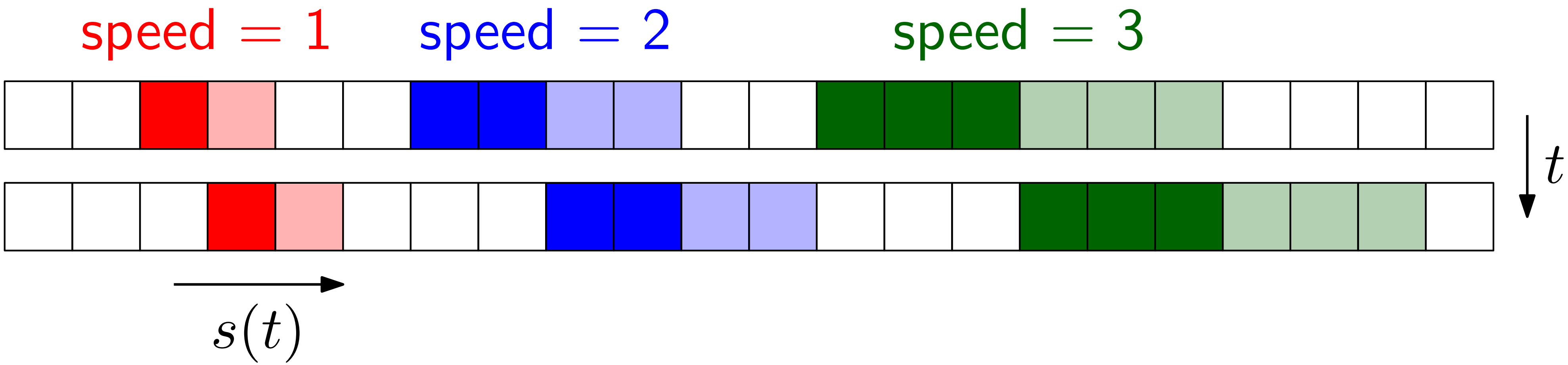

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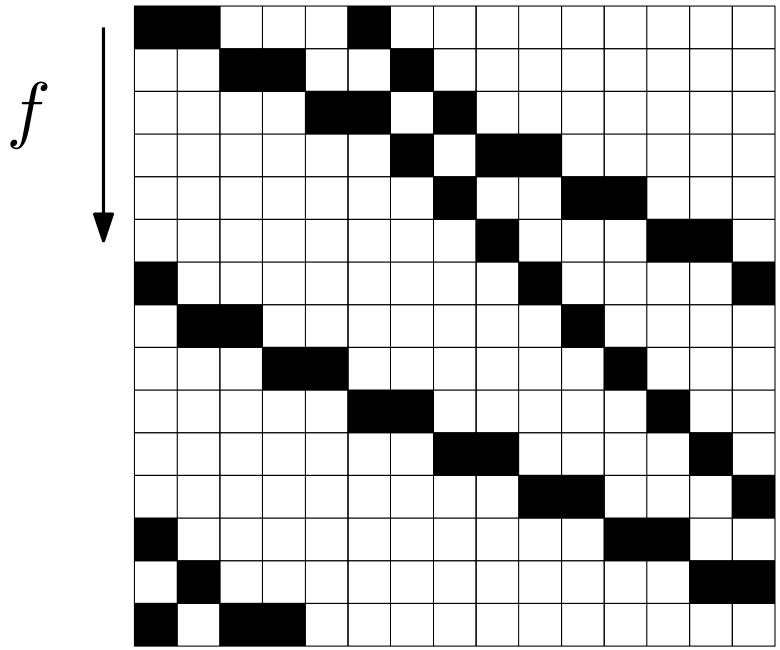
Overtaking of gliders

$(n, k) = (15, 4)$



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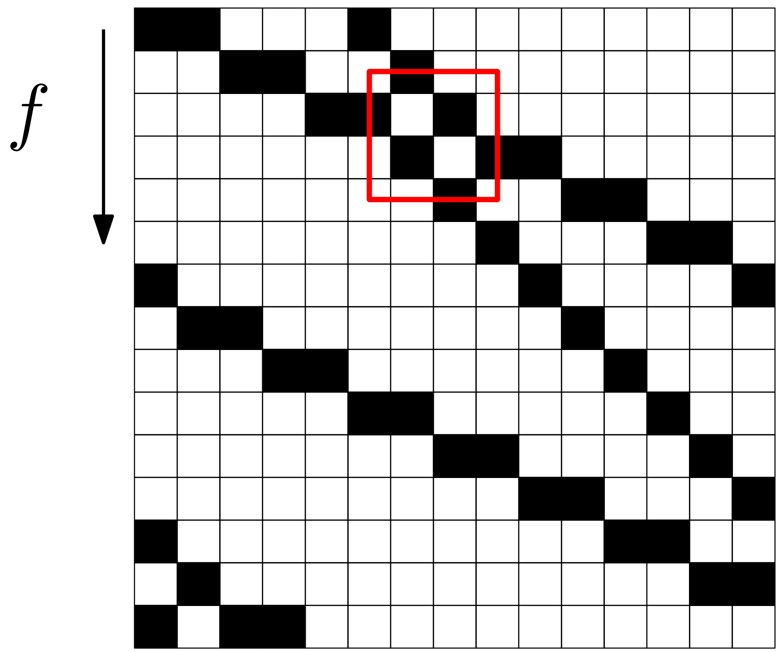

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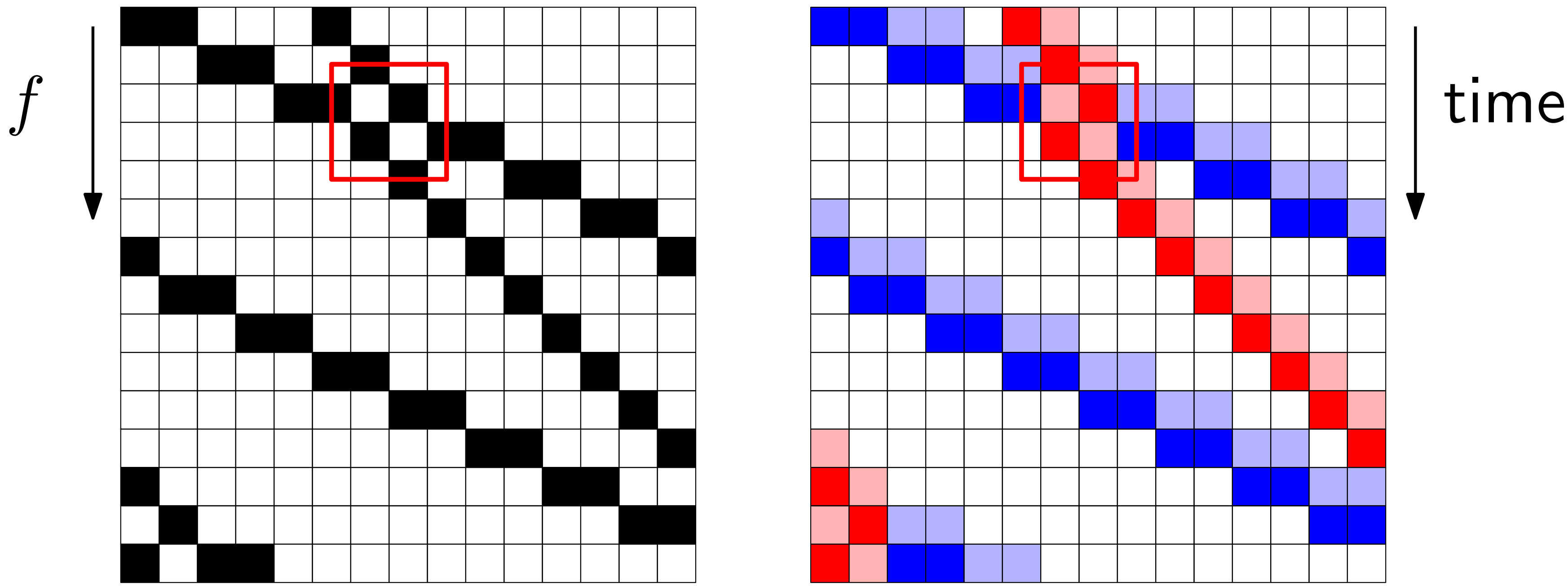

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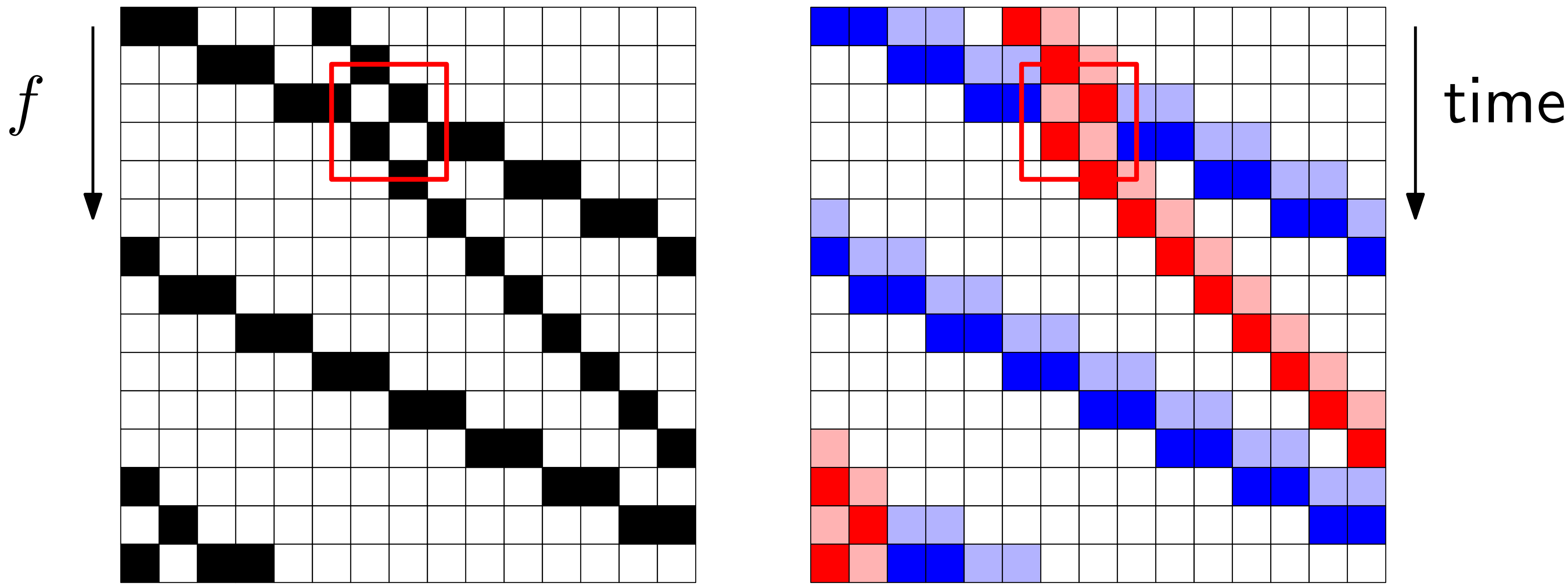

- during overtaking, slower glider stands still for two time steps

Overtaking of gliders

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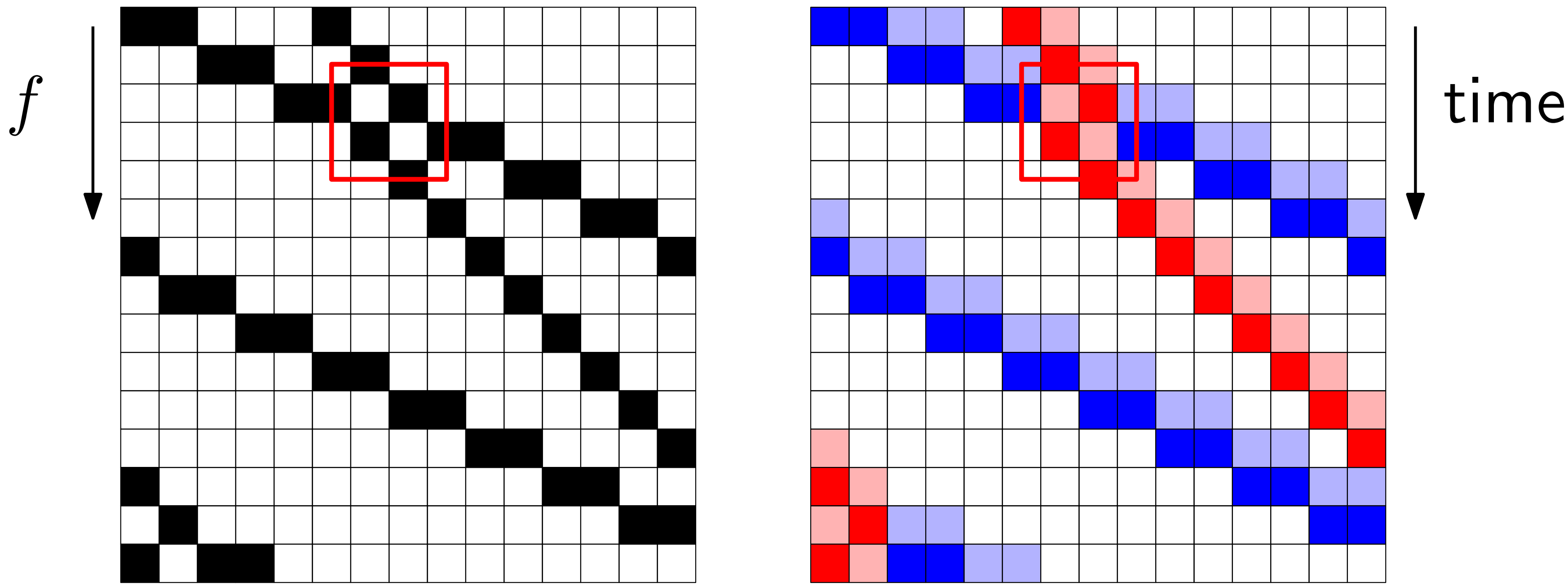

- during overtaking, slower glider stands still for two time steps
- faster glider is boosted by twice the speed of slower glider

Overtaking of gliders

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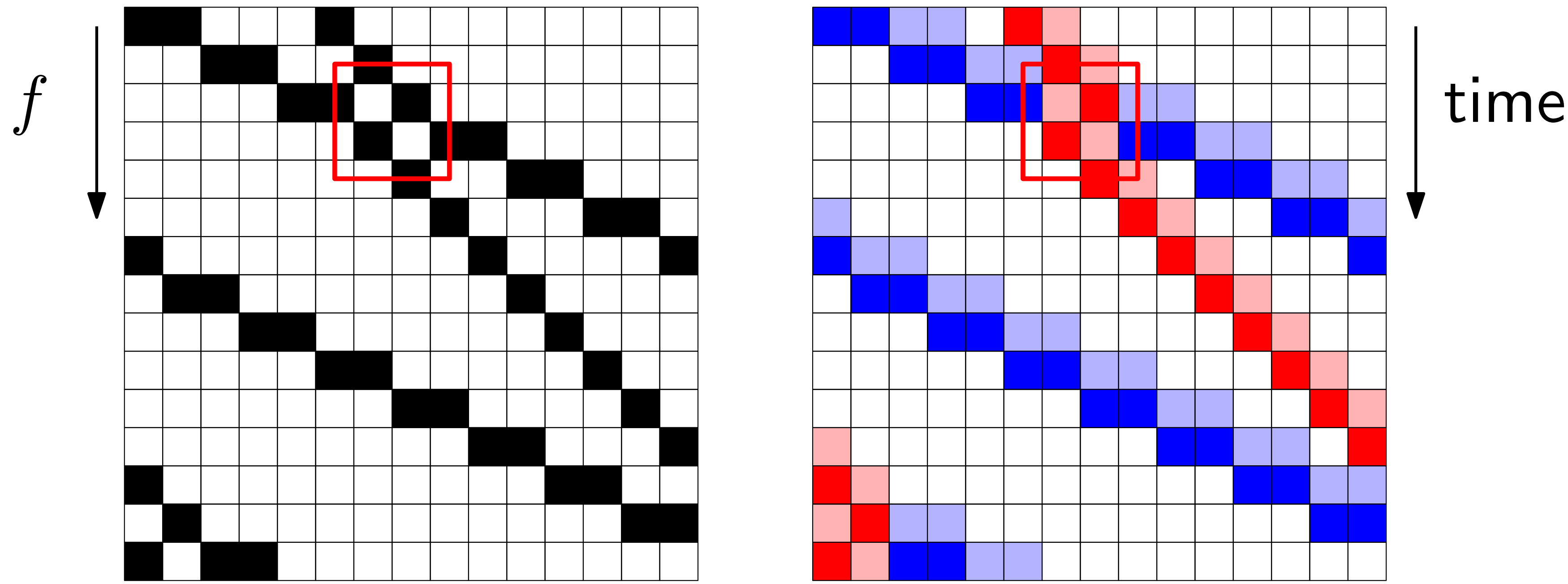
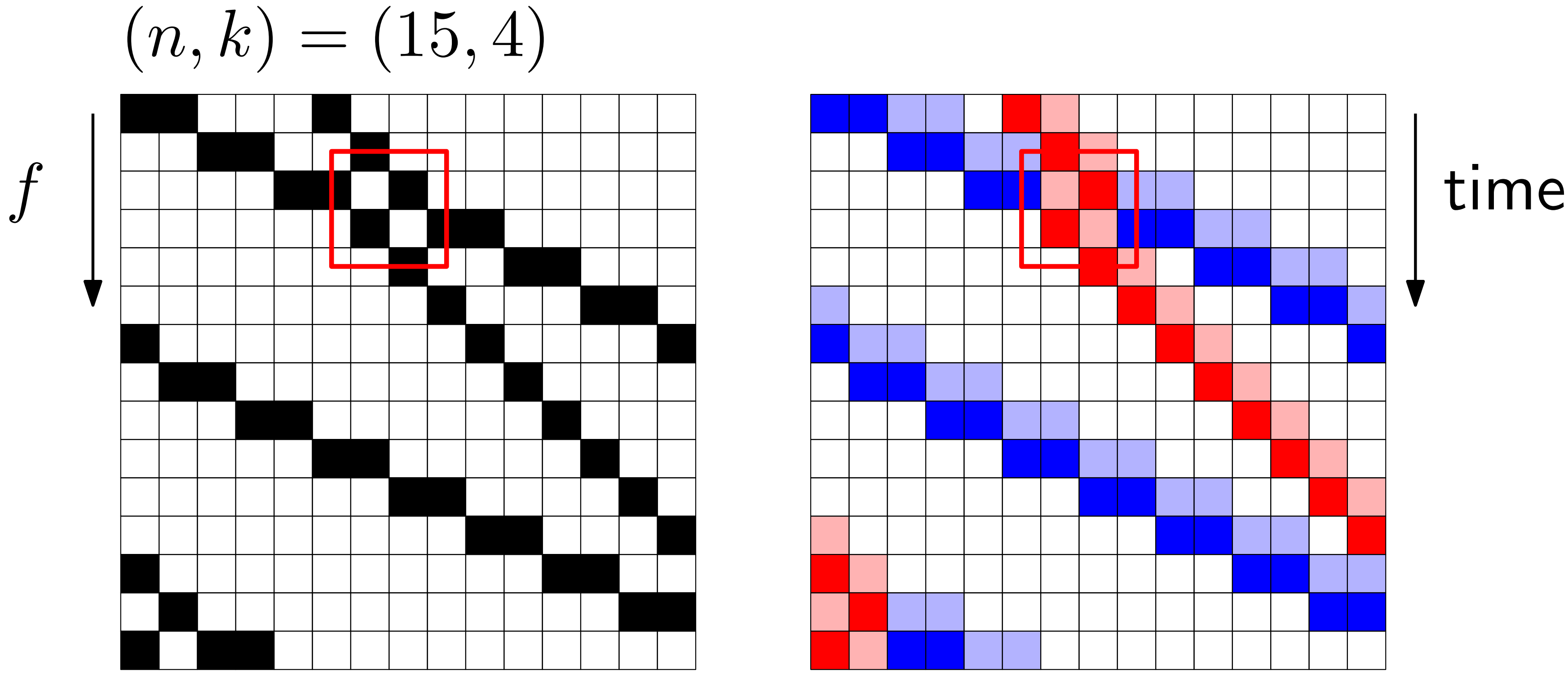


- non-uniform equations of motion:

$$s_1(t) = v_1 \cdot t + s_1(0)$$
$$s_2(t) = v_2 \cdot t + s_2(0)$$

Overtaking of gliders



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Overtaking of gliders

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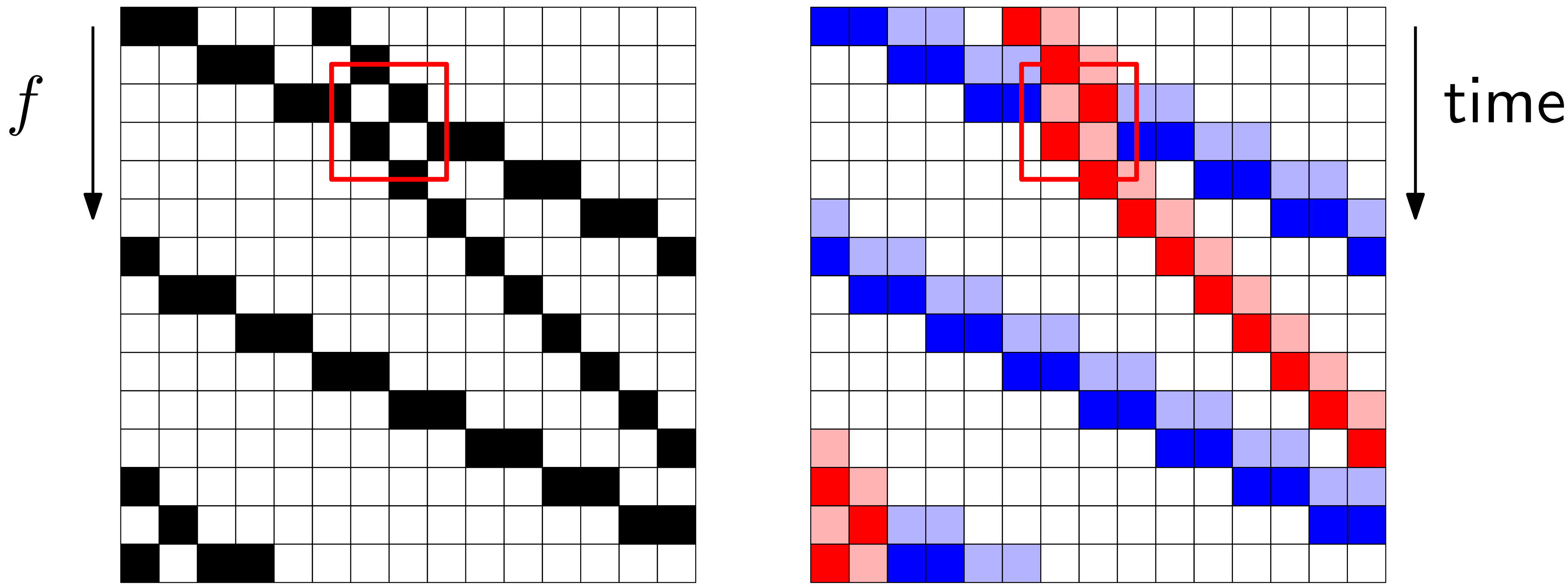

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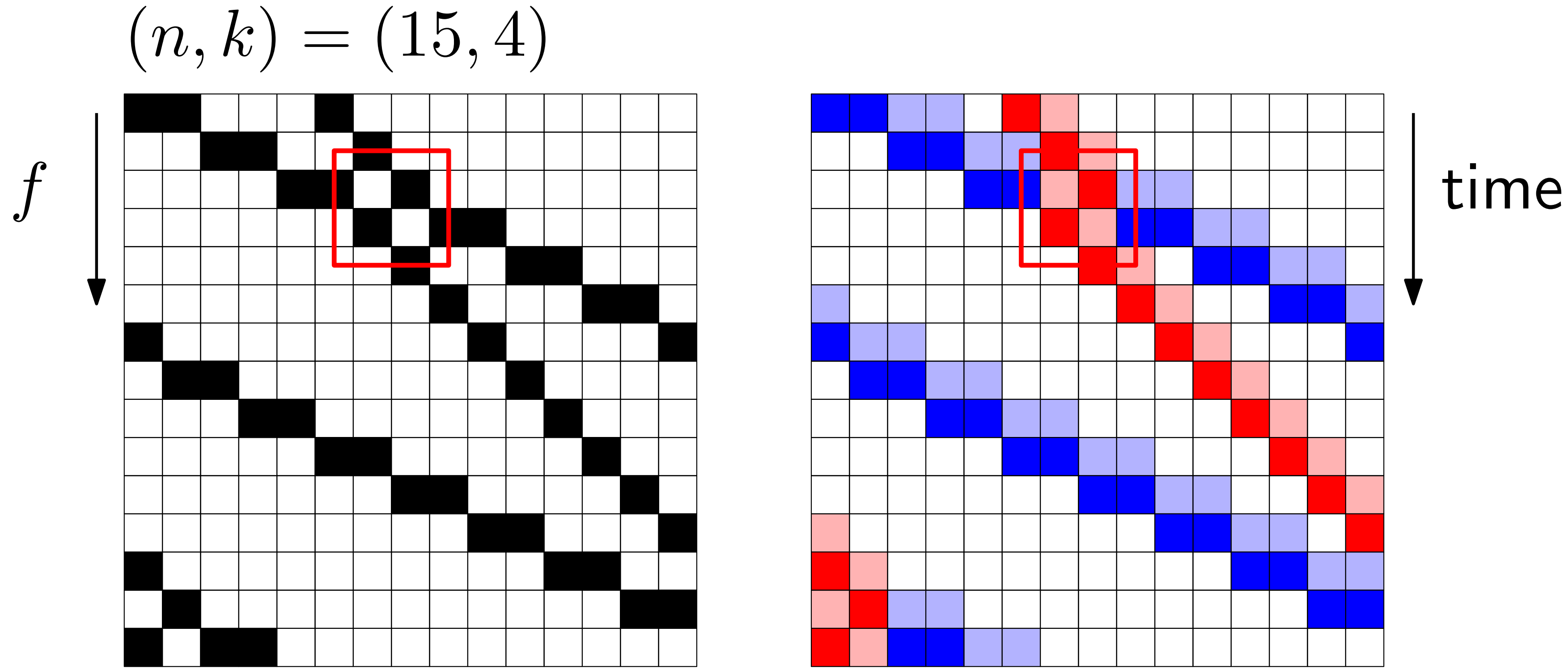
$c_{1,2}$:= number of overtakings

Overtaking of gliders



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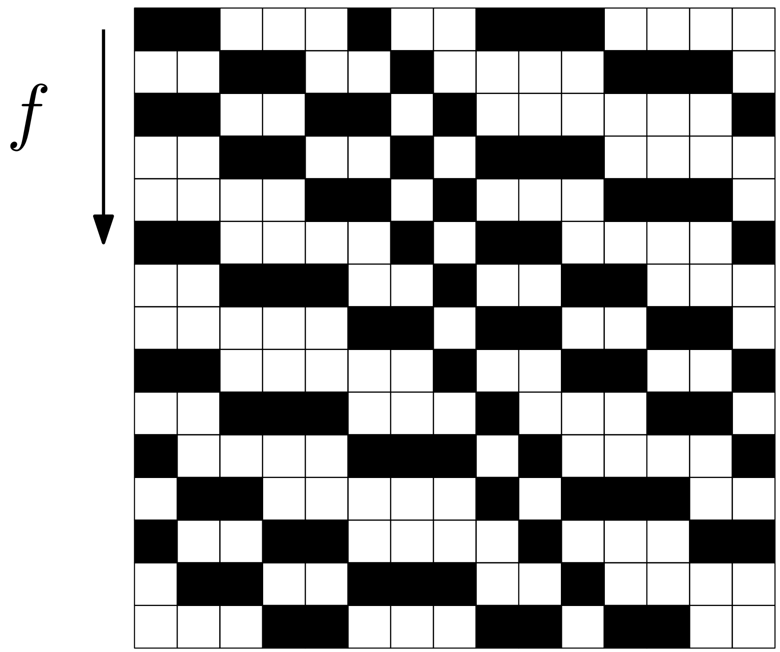
energy conservation!

$c_{1,2} :=$ number of overtakings

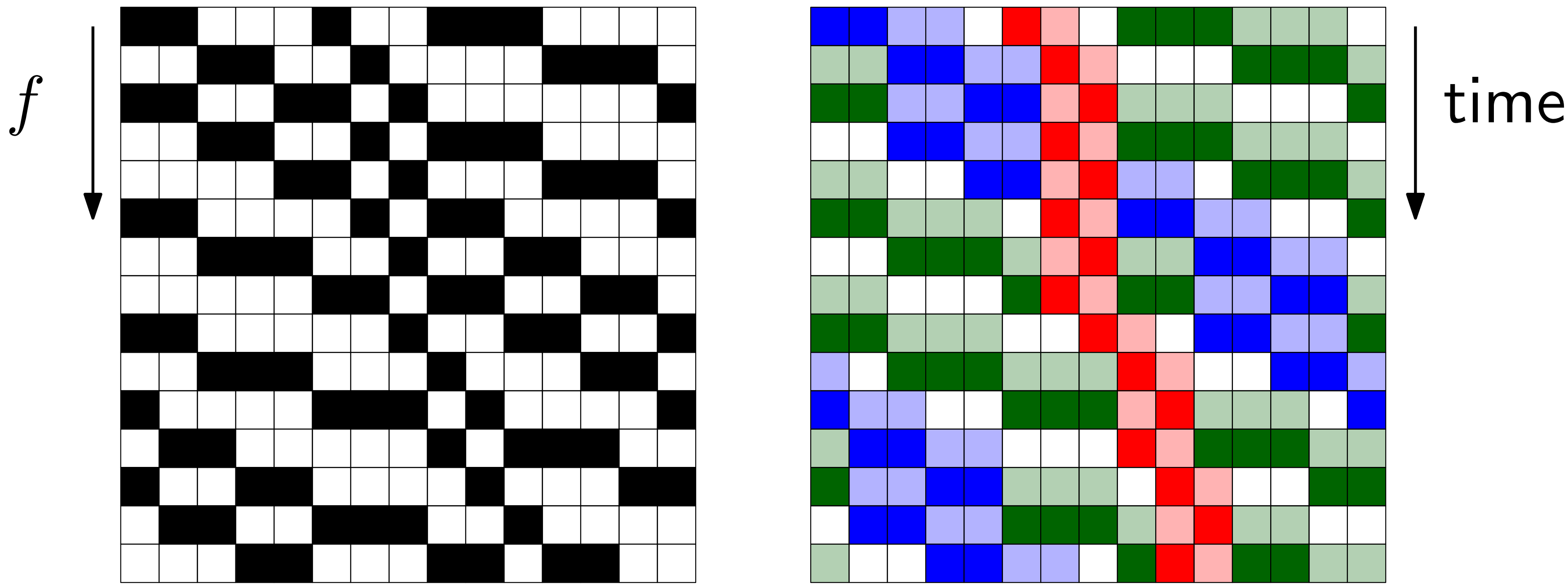
Glider partition

$(n, k) = (15, 6)$



Glider partition

$(n, k) = (15, 6)$



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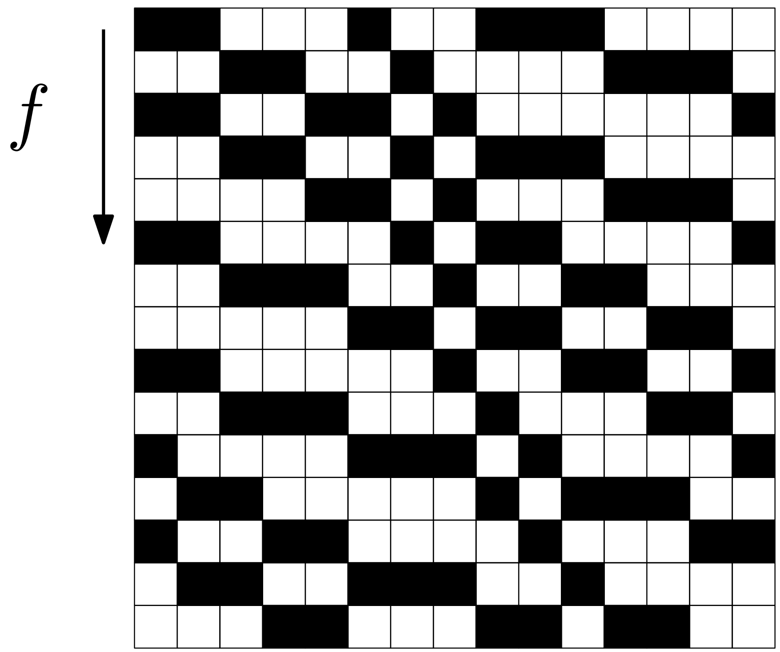

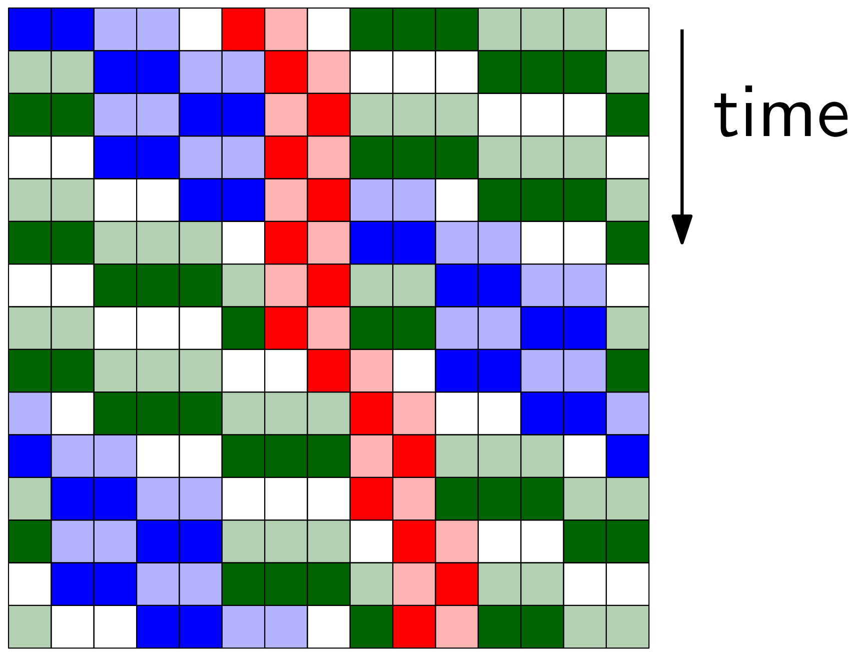

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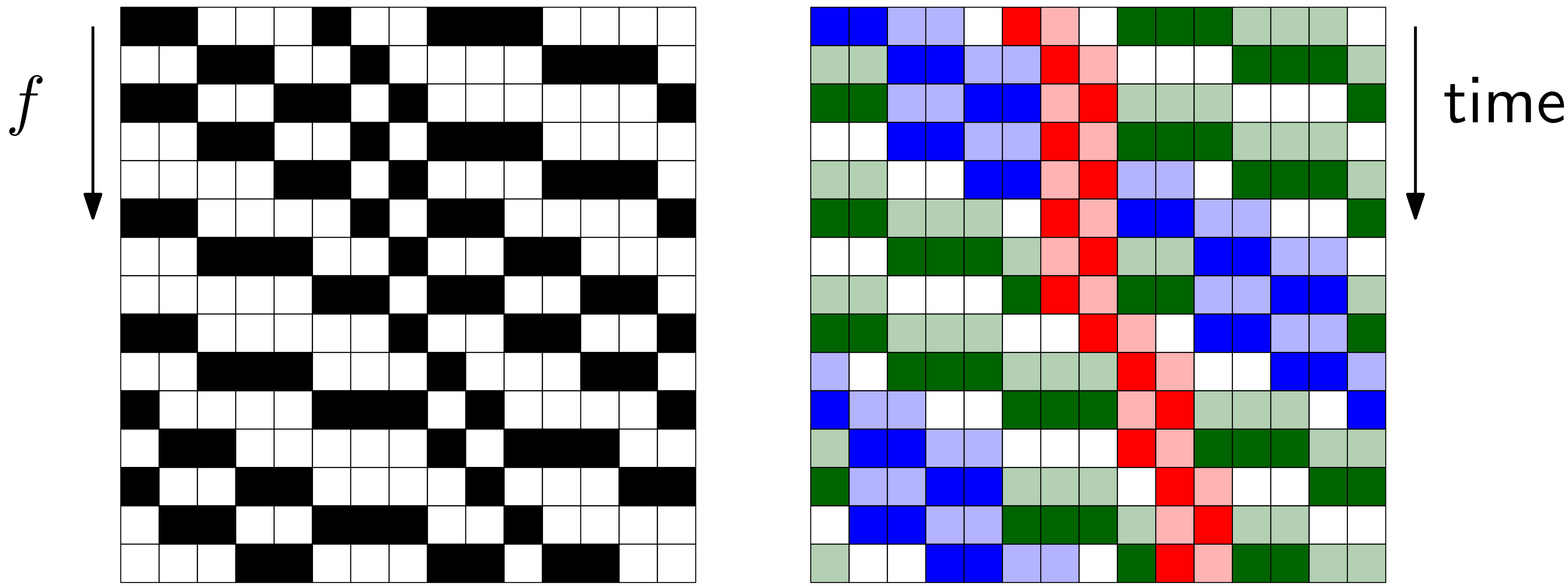

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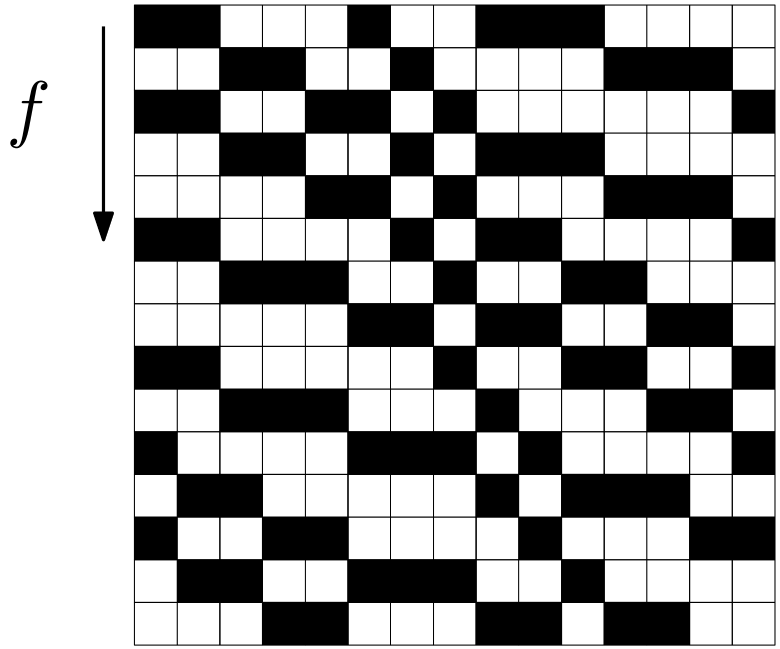

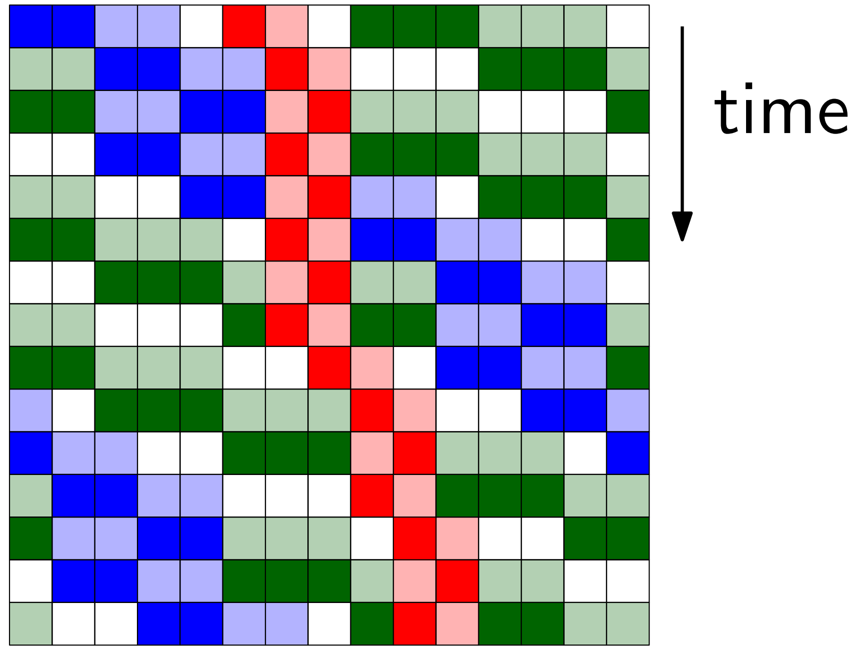

- gliders can be interleaved in complicated ways
- general glider partition rule works recursively on Motzkin path
- general equations of motion have overtaking counters $c_{i,j}$ for all pairs of gliders i, j

Cycle invariant

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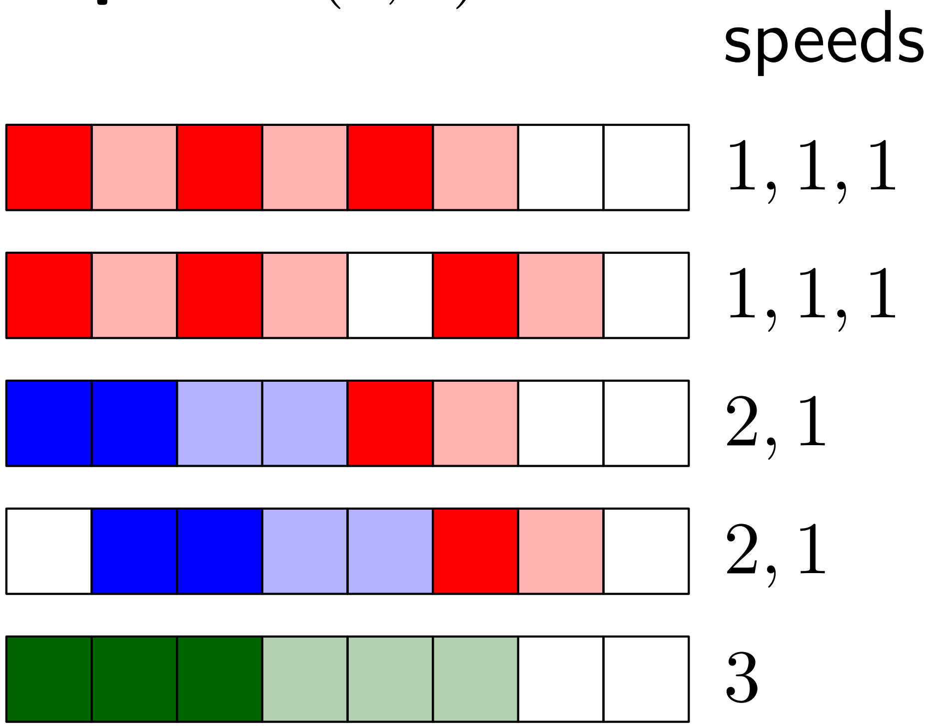

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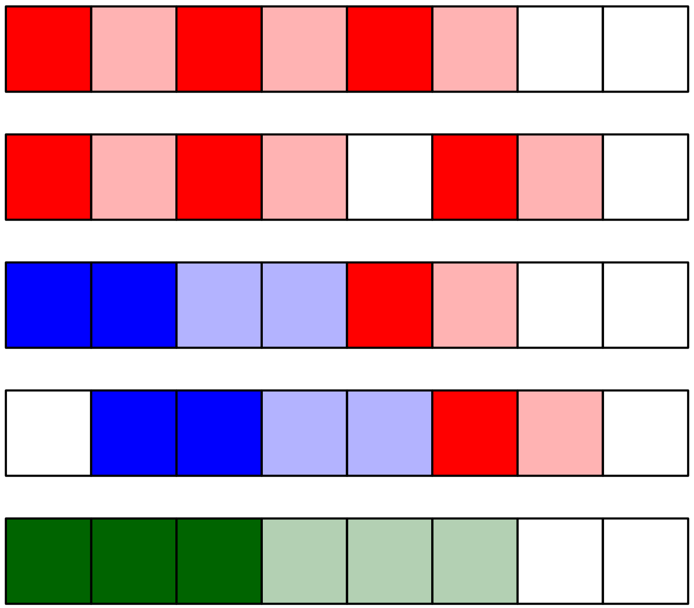

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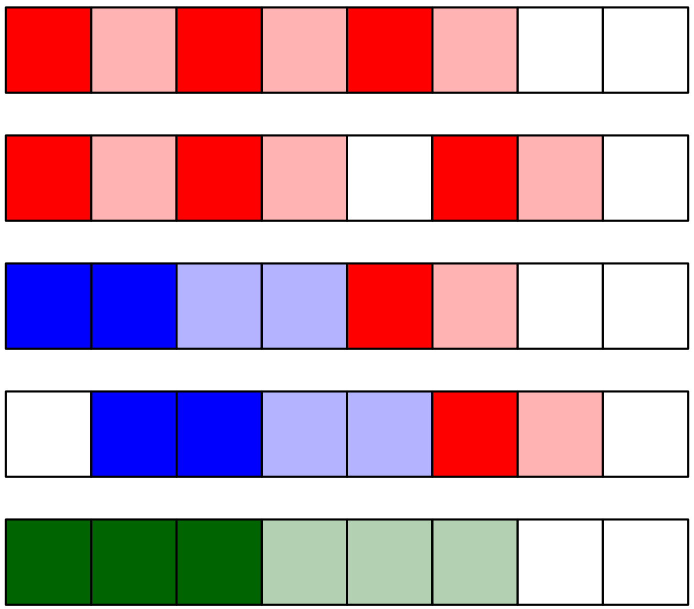

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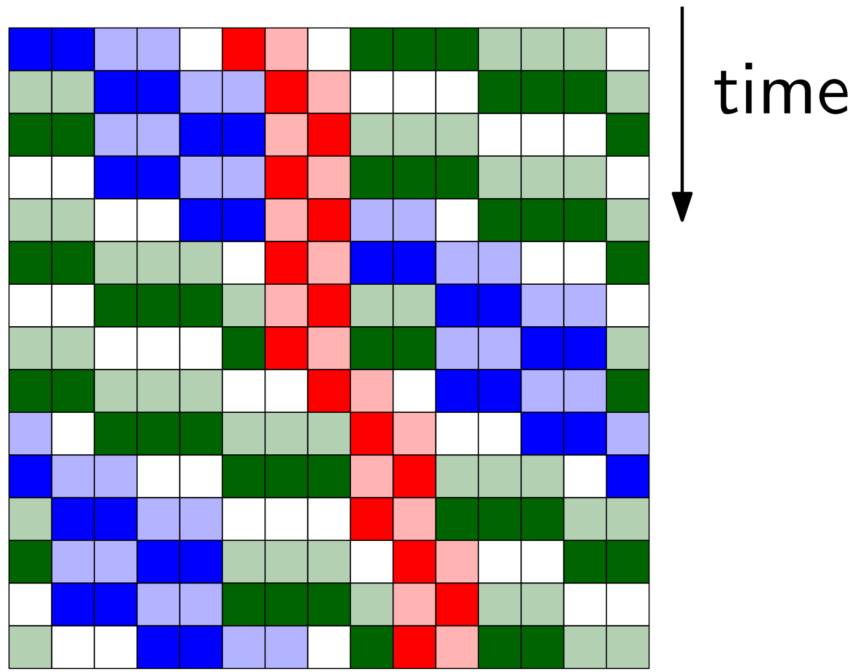

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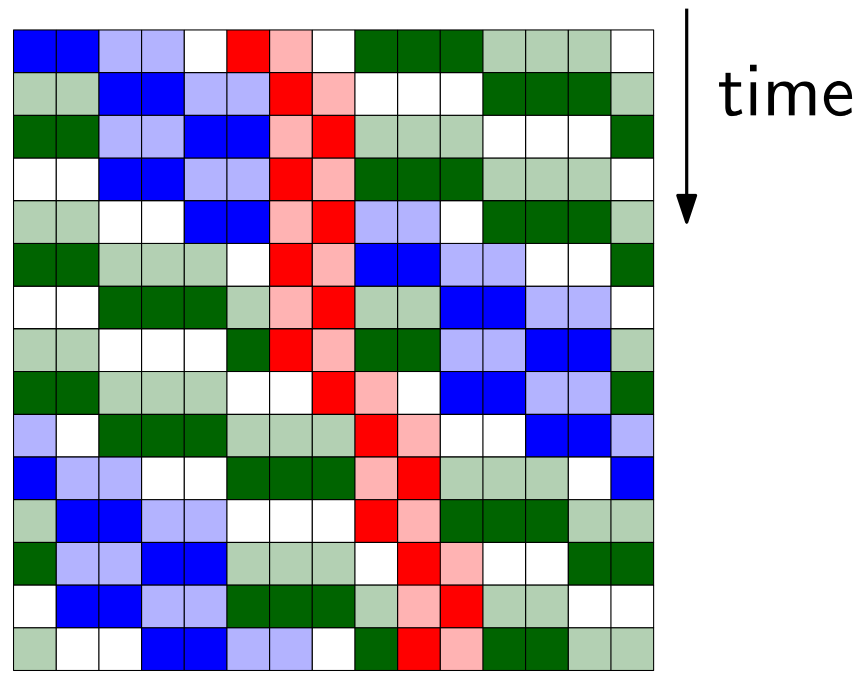
- **Lemma:** For any cycle in $K(n,k)$ and every glider, there is $t > 0$ such that $s(t) > s(0)$.



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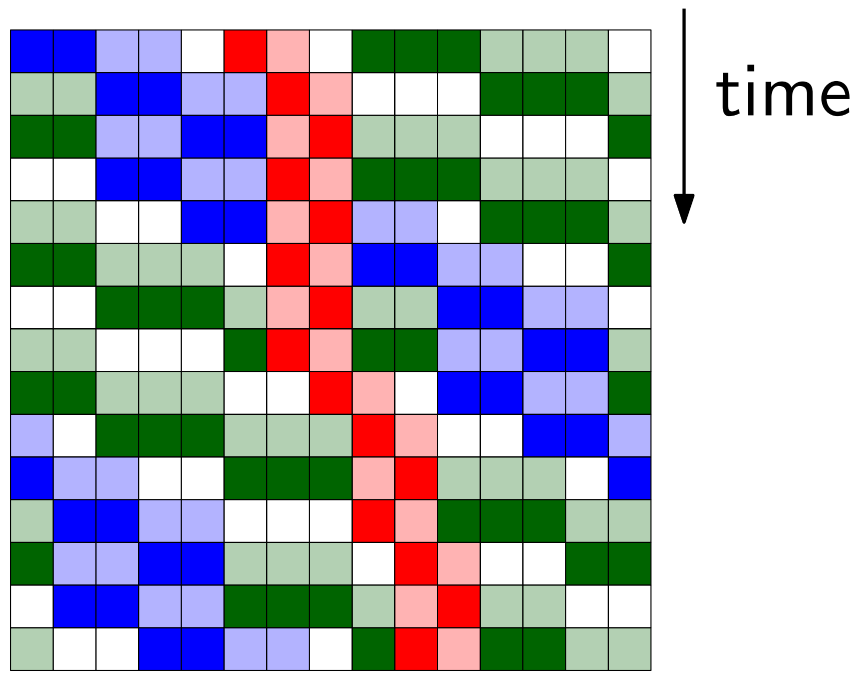

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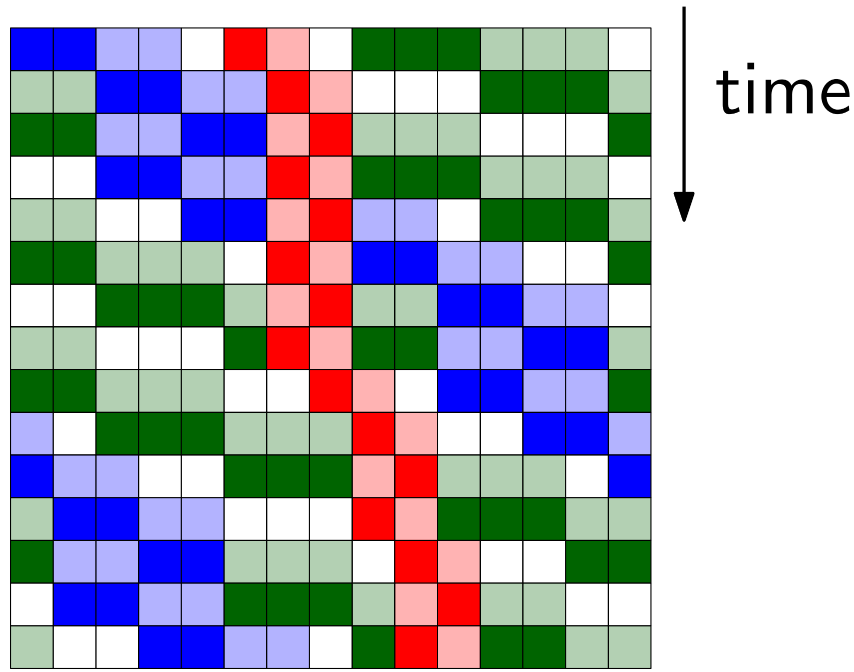
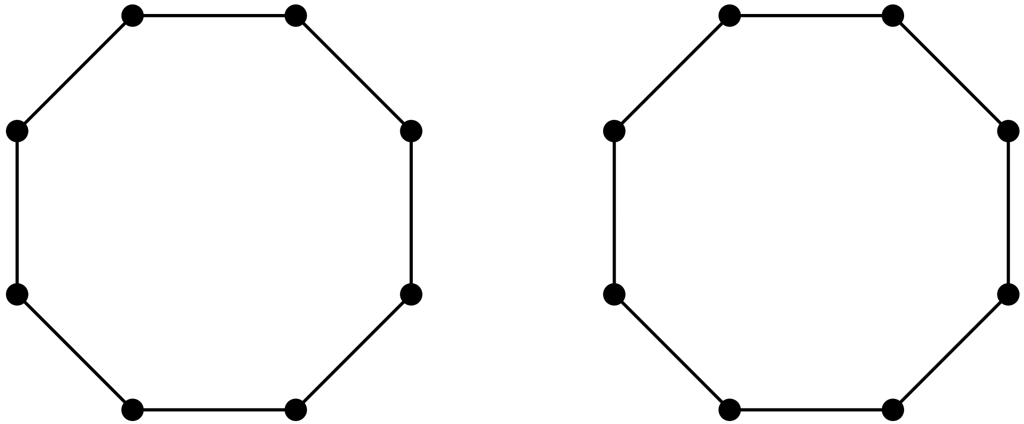
- proved by showing that matrix of equations of motion is non-singular ($\det \neq 0$).



Gluing cycles



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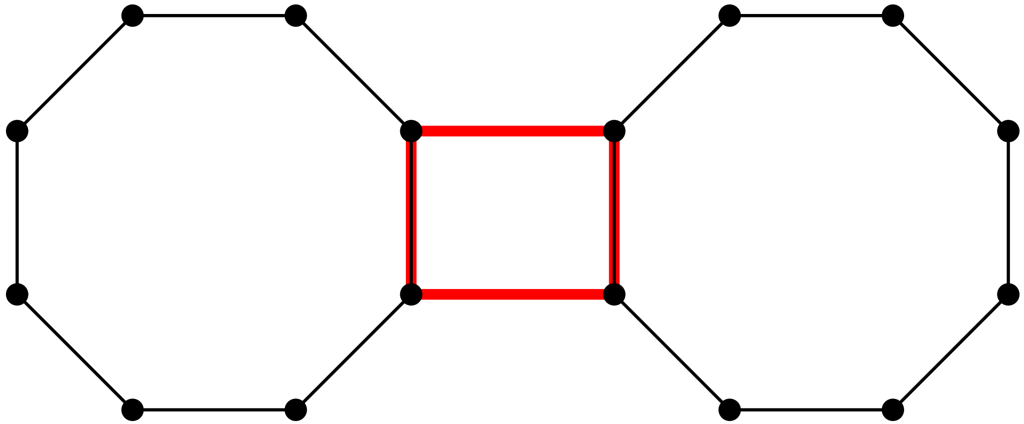

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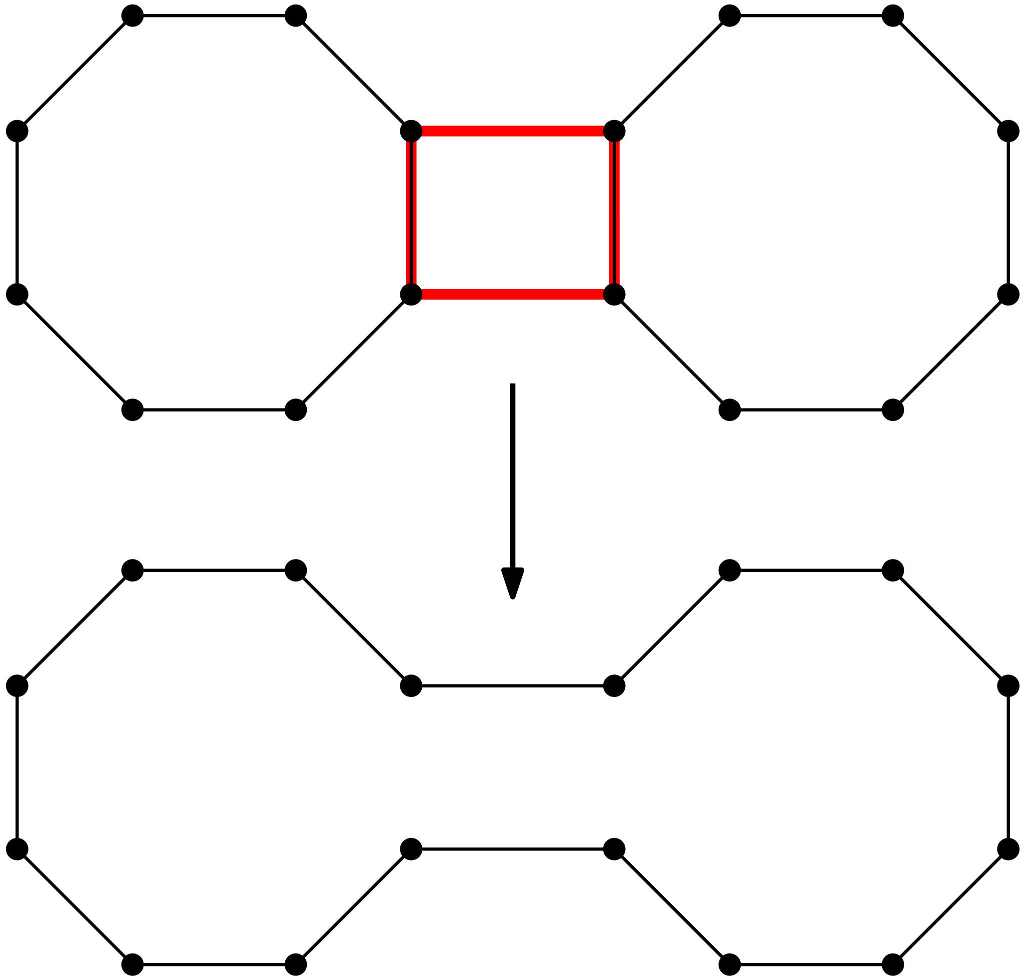

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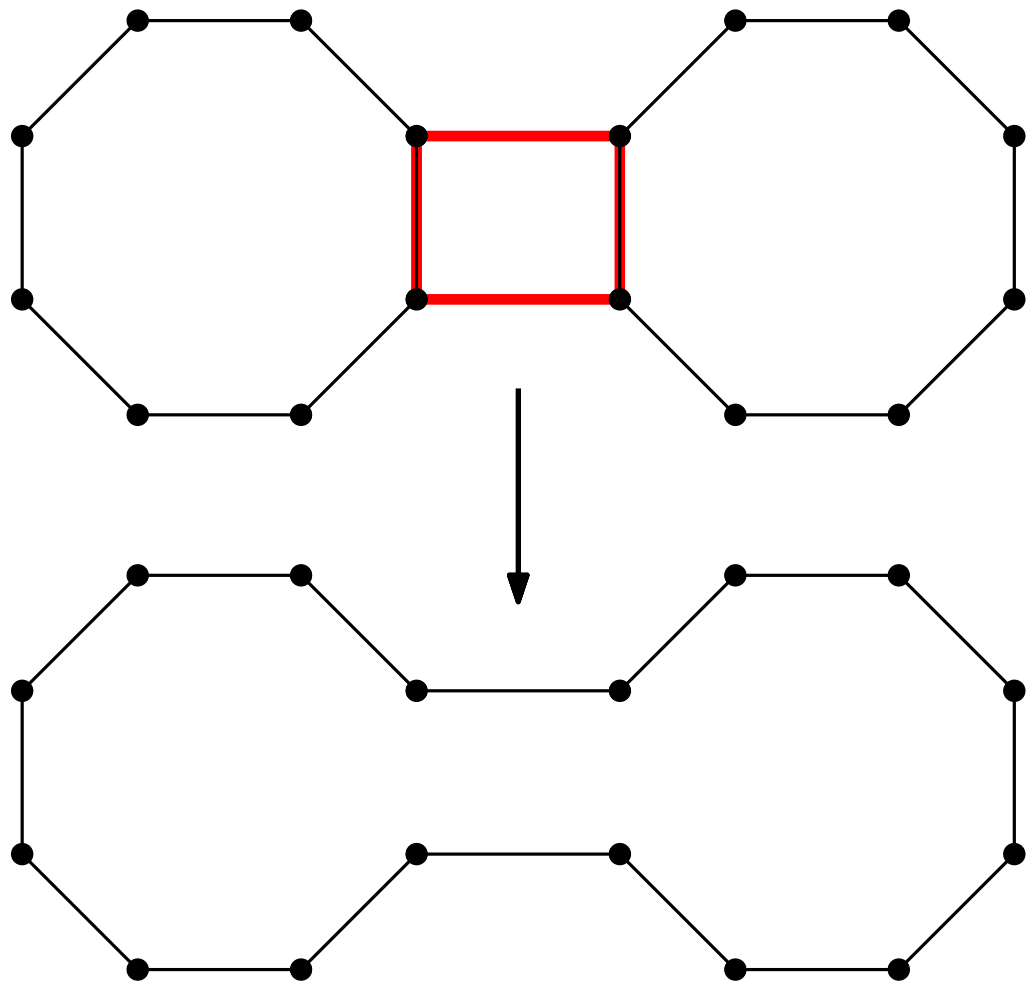
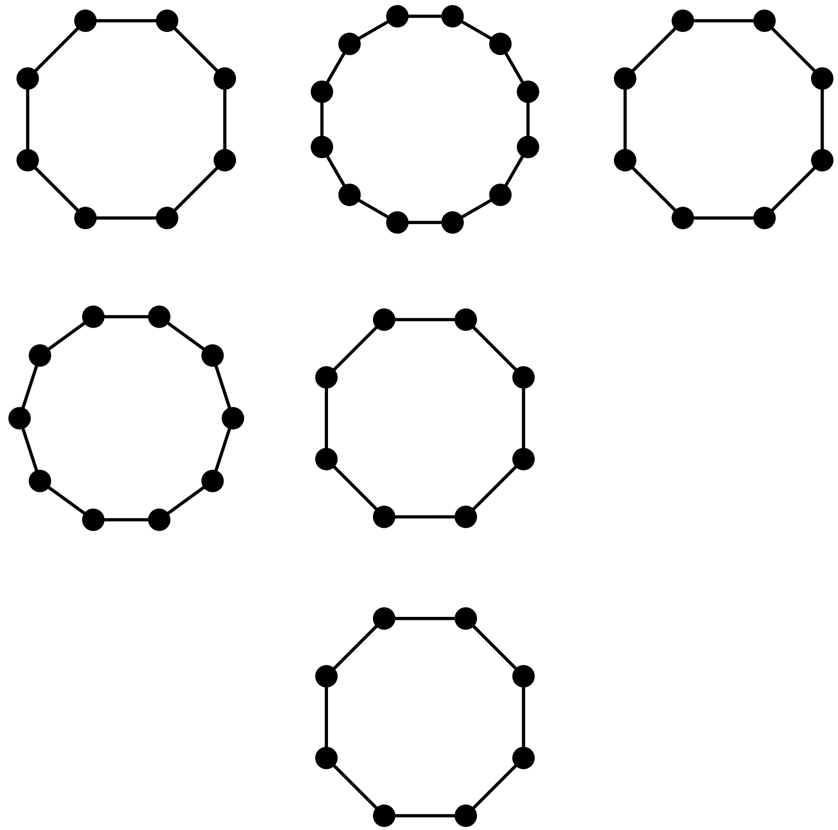


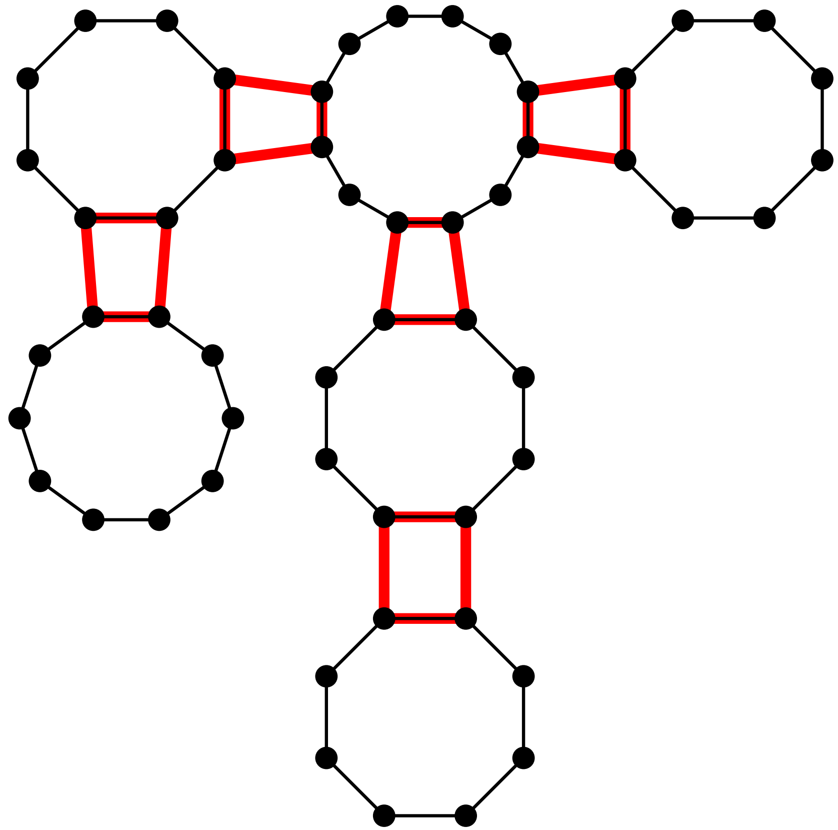
4-cycles exist as $n \geq 2k+3$

Gluing cycles



Gluing cycles

- connect cycles of factor to a single Hamilton cycle (tree-like)



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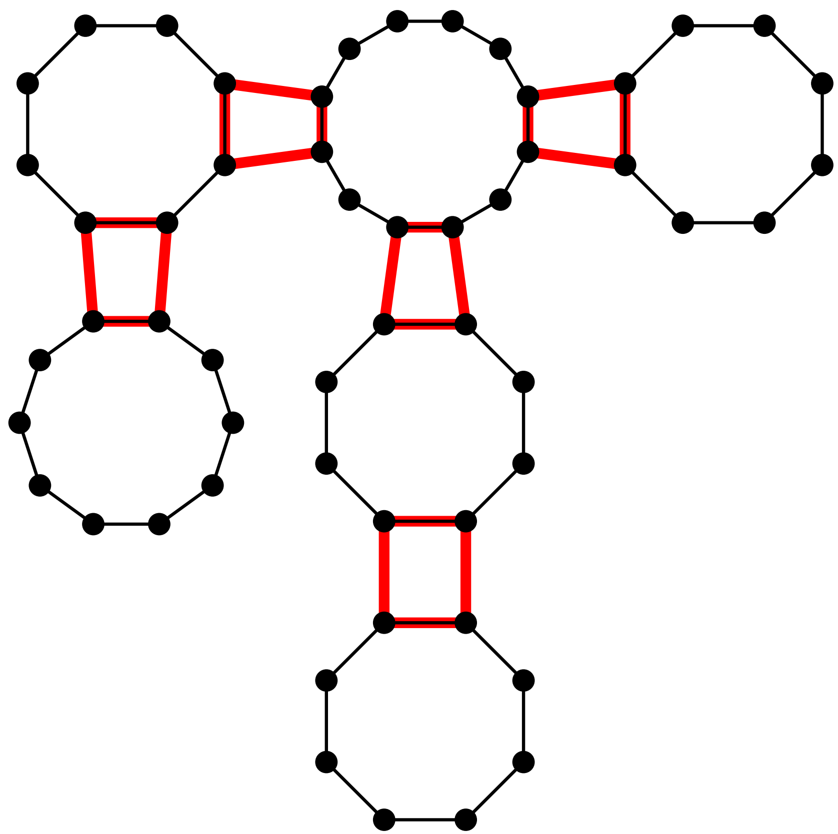

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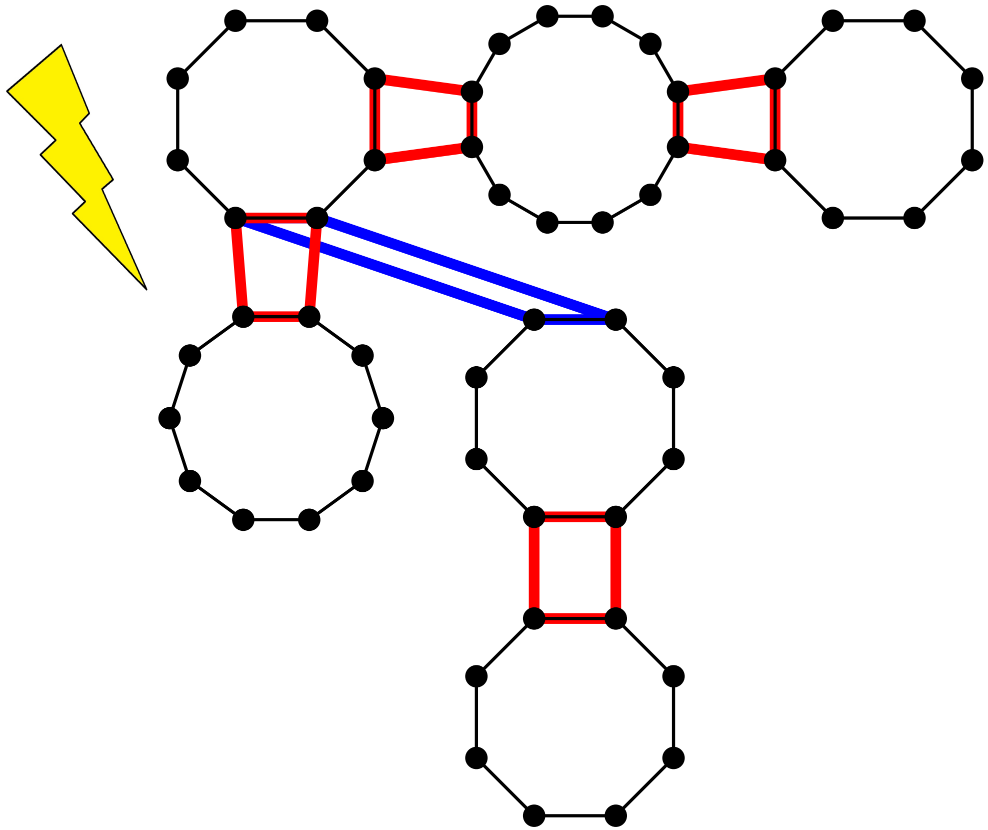

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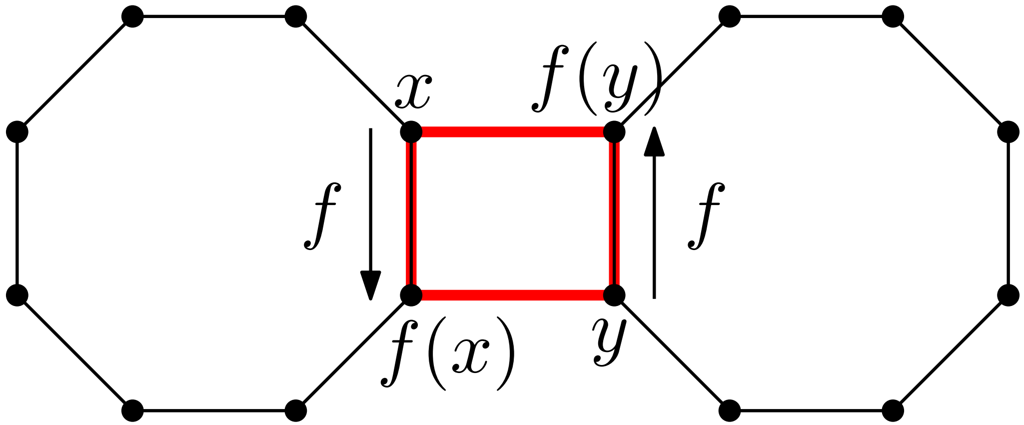

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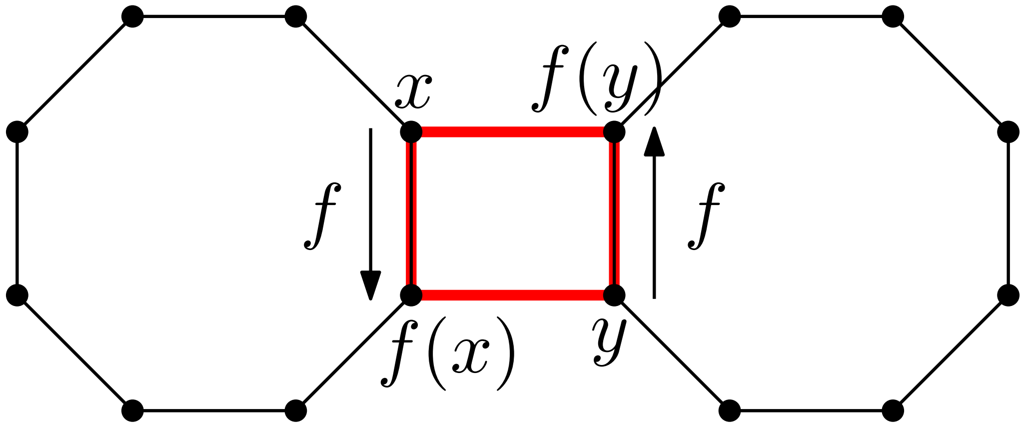

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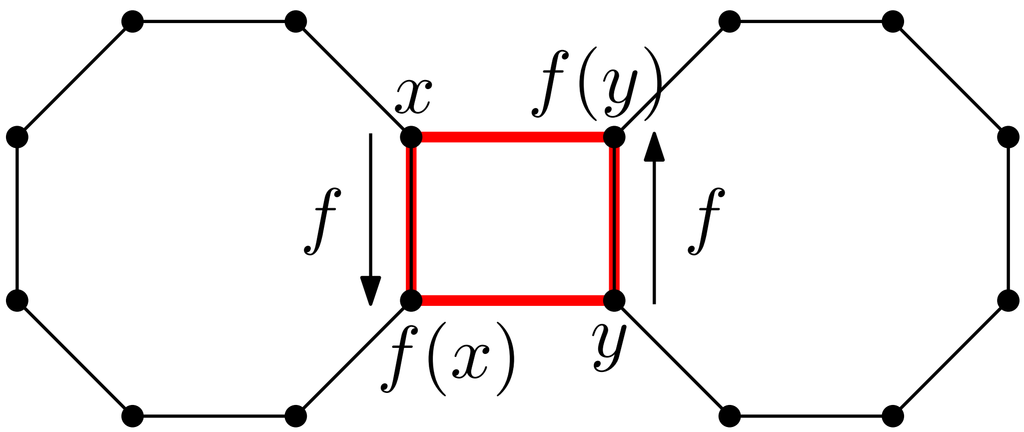

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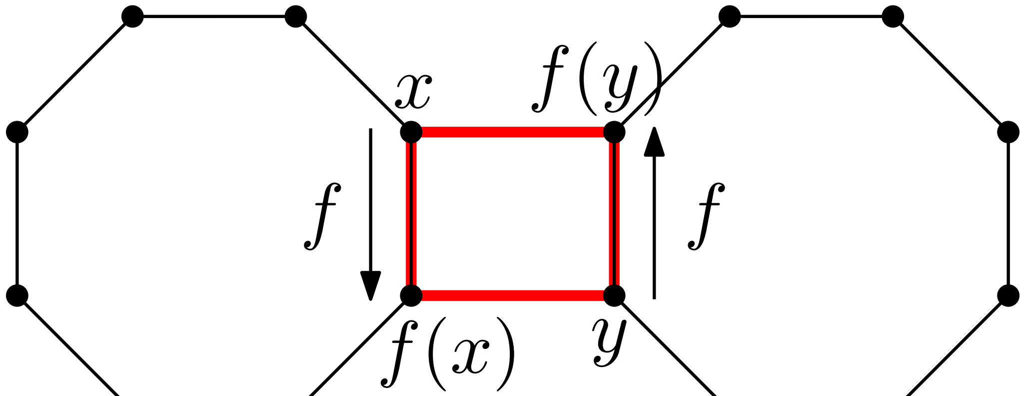

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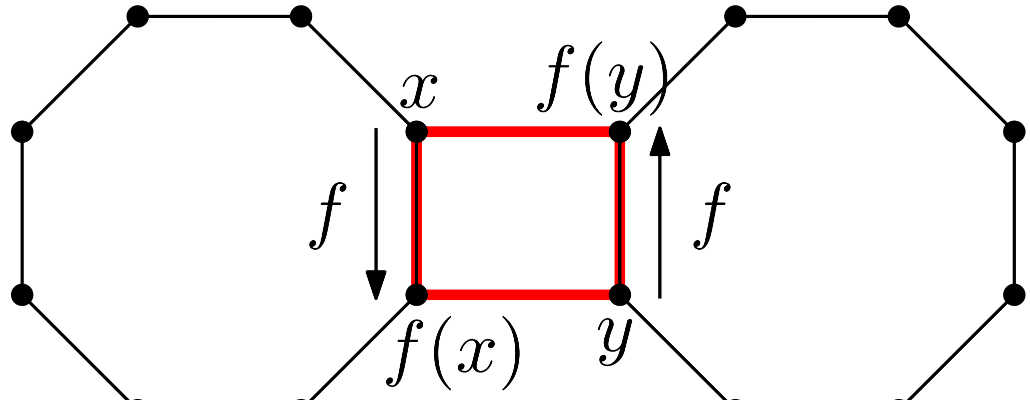

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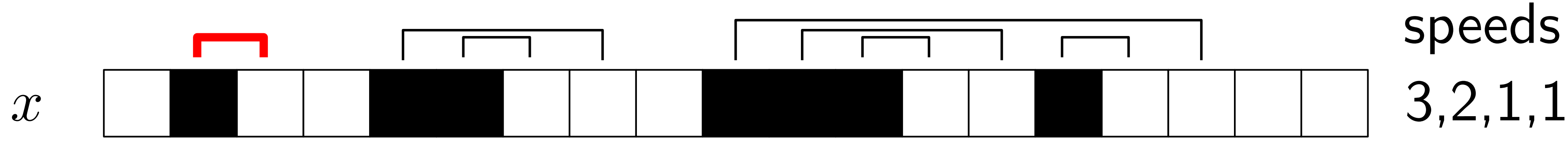

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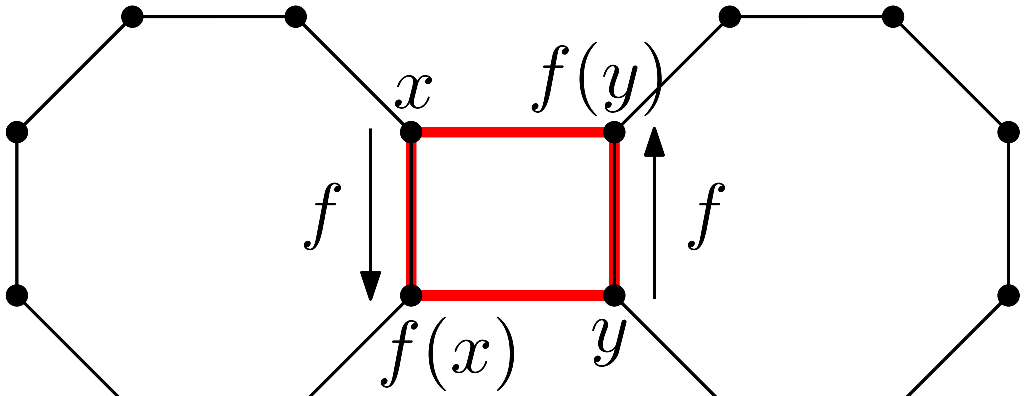

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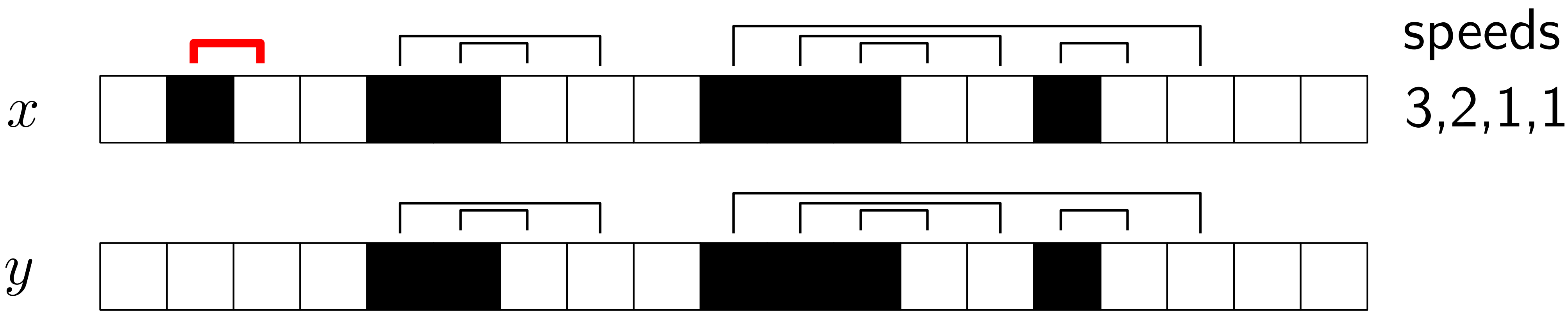

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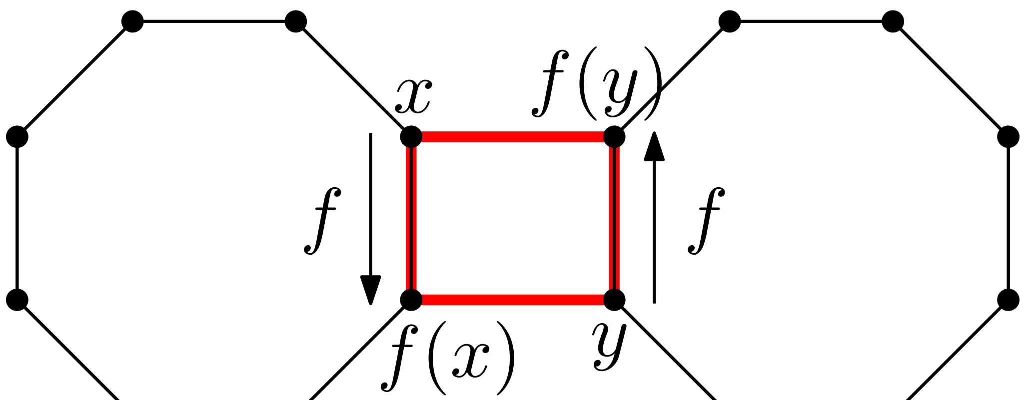

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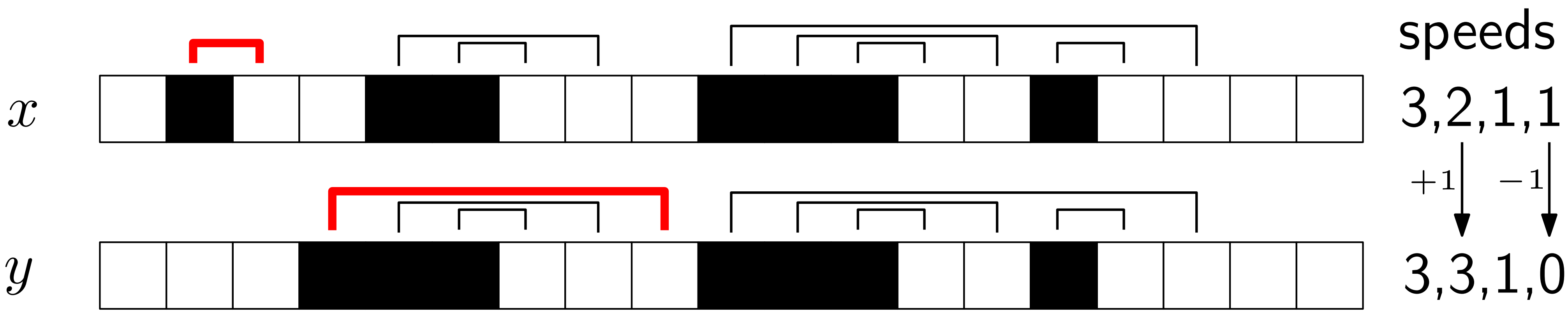

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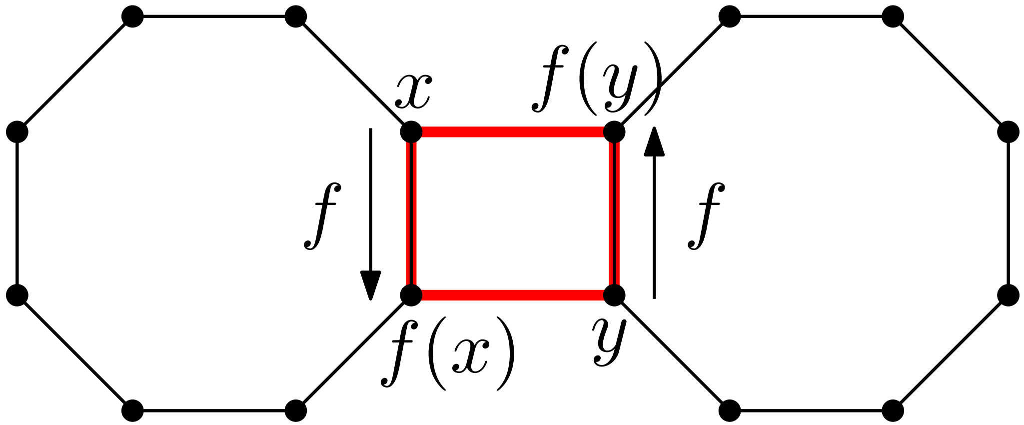

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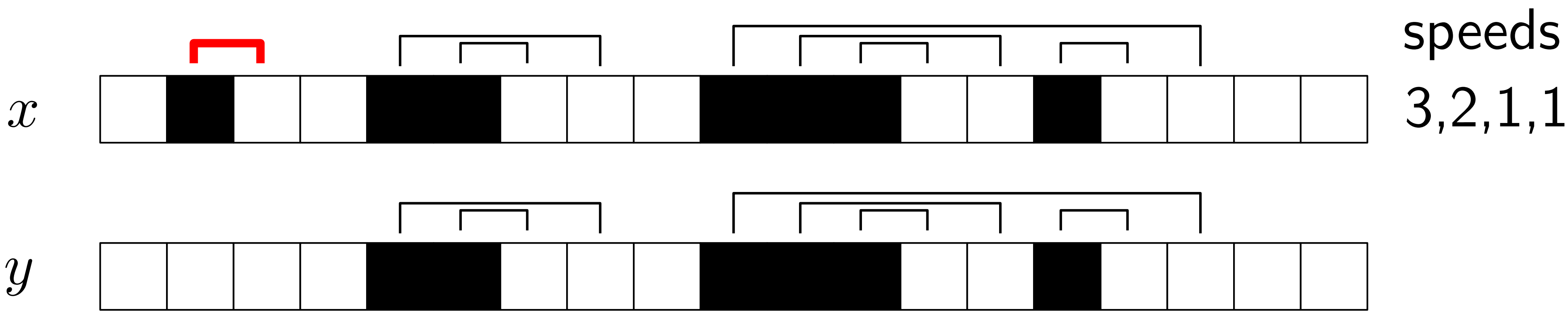

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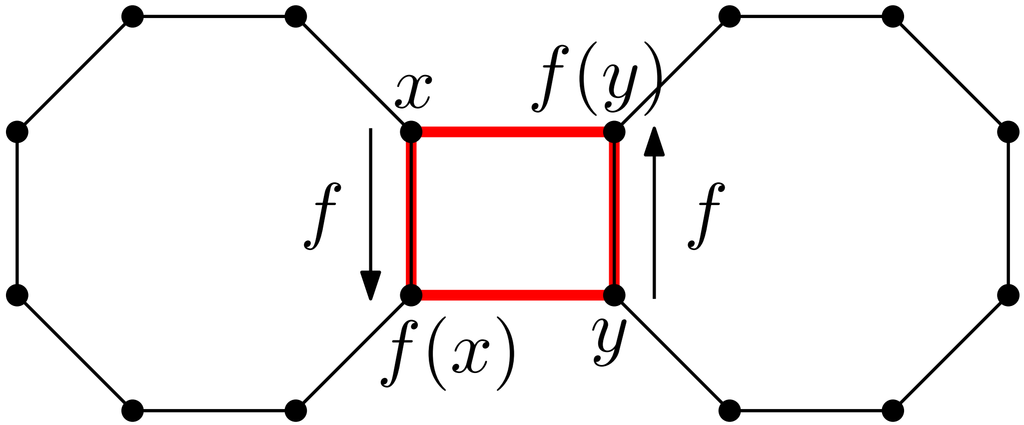

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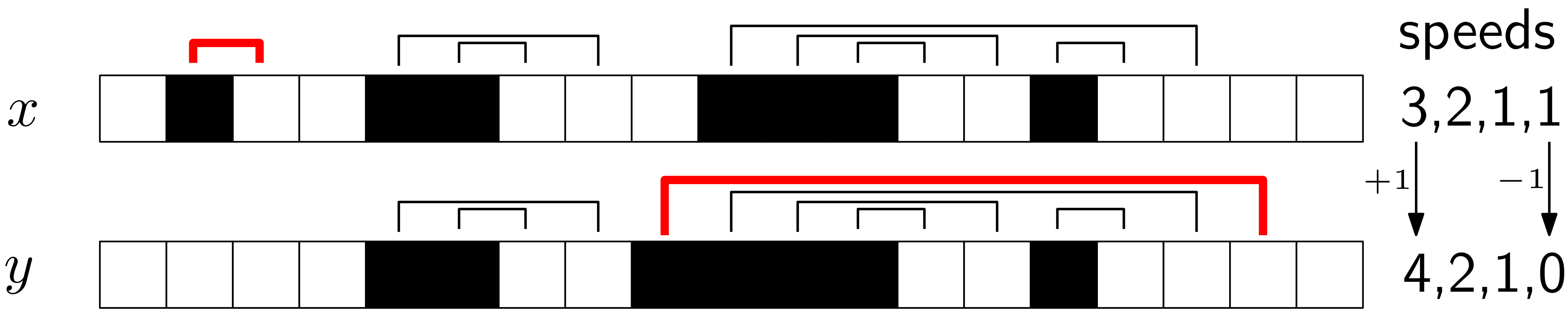

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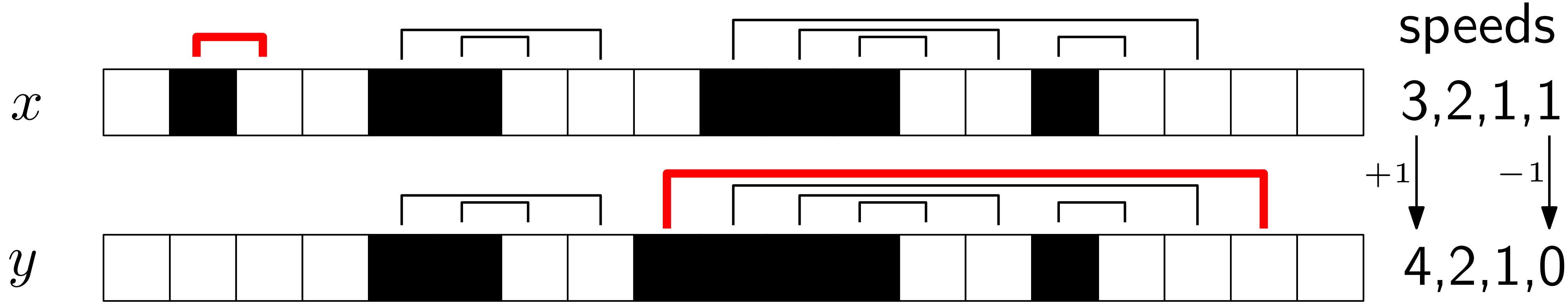

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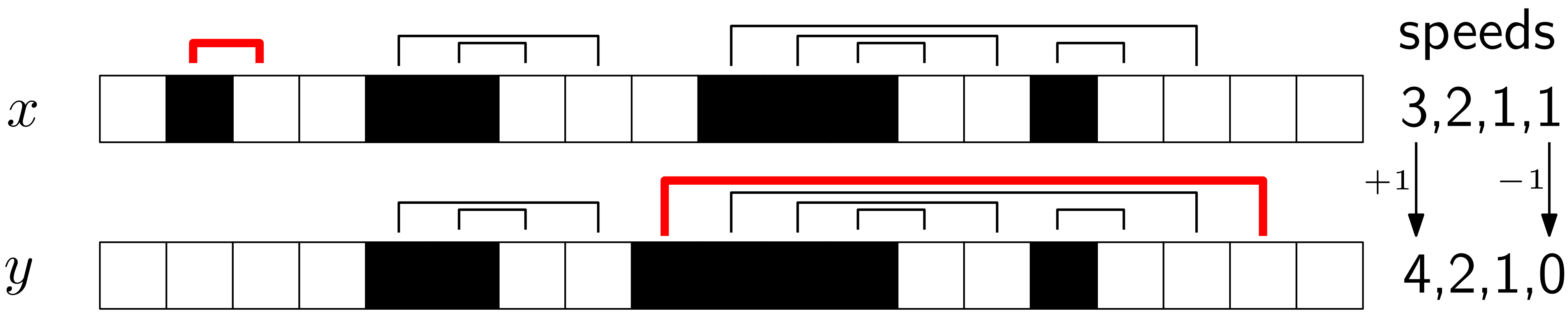

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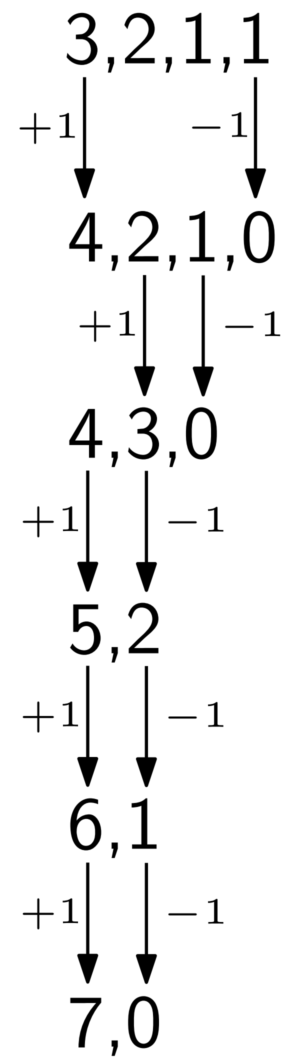

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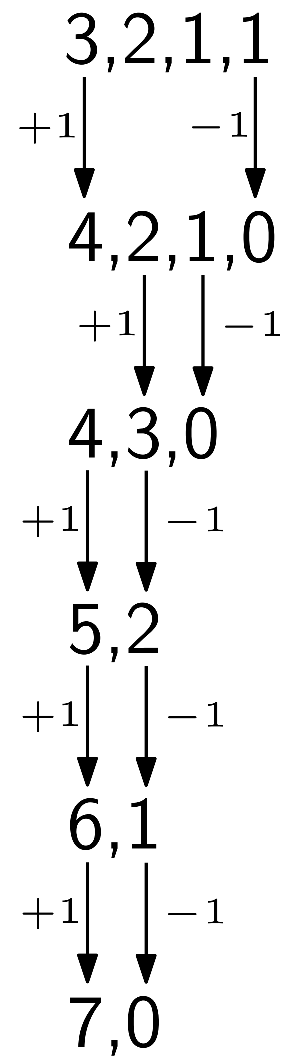

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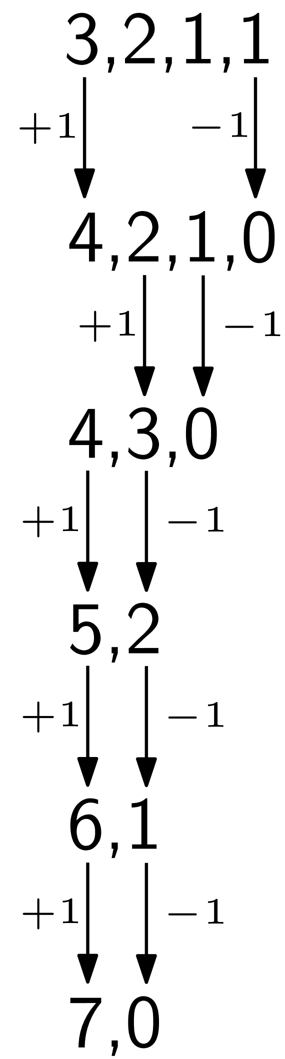
- proves connectivity



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Thank you!